

A CONVENIENT COORDINATIZATION OF SIEGEL-JACOBI DOMAINS

STEFAN BERCEANU

ABSTRACT. We determine the homogeneous Kähler diffeomorphism FC which expresses the Kähler two-form on the Siegel-Jacobi ball $\mathcal{D}_n^J = \mathbb{C}^n \times \mathcal{D}_n$ as the sum of the Kähler two-form on \mathbb{C}^n and the one on the Siegel ball \mathcal{D}_n . The classical motion and quantum evolution on \mathcal{D}_n^J determined by a hermitian linear Hamiltonian in the generators of the Jacobi group $G_n^J = H_n \rtimes \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ are described by a matrix Riccati equation on \mathcal{D}_n and a linear first order differential equation in $z \in \mathbb{C}^n$, with coefficients depending also on $W \in \mathcal{D}_n$. H_n denotes the $(2n+1)$ -dimensional Heisenberg group. The system of linear differential equations attached to the matrix Riccati equation is a linear Hamiltonian system on \mathcal{D}_n . When the transform $FC : (\eta, W) \rightarrow (z, W)$ is applied, the first order differential equation in the variable $\eta = (\mathbb{I}_n - W\bar{W})^{-1}(z + W\bar{z}) \in \mathbb{C}^n$ becomes decoupled from the motion on the Siegel ball. Similar considerations are presented for the Siegel-Jacobi upper half plane $\mathcal{X}_n^J = \mathbb{C}^n \times \mathcal{X}_n$, where \mathcal{X}_n denotes the Siegel upper half plane.

CONTENTS

1. Introduction	2
2. The Jacobi algebra \mathfrak{g}_n^J	4
3. Coherent states on \mathcal{D}_n^J	6
4. The differential action	9
5. The real Jacobi group $G_n^J(\mathbb{R})$	13
5.1. Comparison with Yang's results	15
6. The fundamental conjecture for the Siegel-Jacobi domains	16
7. Classical motion and quantum evolution on Siegel-Jacobi domains	19
7.1. Equations of motion	20
7.2. Solution of the equations of motion	23
8. Phases	28
8.1. Berry phase for \mathcal{D}_n^J	28
8.2. Dynamical phase	28
References	29

2010 *Mathematics Subject Classification.* 81R30, 32Q15, 81V80, 81S10, 34A05.

Key words and phrases. Jacobi group, coherent and squeezed states, Siegel-Jacobi domains, fundamental conjecture for homogeneous Kähler manifolds, matrix Riccati equation, Berezin quantization.

1. INTRODUCTION

The Jacobi groups [26] are semidirect products of appropriate semisimple real algebraic groups of hermitian type with Heisenberg groups [59, 41]. The Jacobi groups are unimodular, nonreductive, algebraic groups of Harish-Chandra type [50]. The Siegel-Jacobi domains are nonsymmetric domains associated to the Jacobi groups by the generalized Harish-Chandra embedding [50, 41], [64] - [66].

The holomorphic irreducible unitary representations of the Jacobi groups based on Siegel-Jacobi domains have been constructed by Berndt, Böcherer, Schmidt, and Takase [19, 20], [57] - [59]. Some coherent state systems based on Siegel-Jacobi domains have been investigated in the framework of quantum mechanics, geometric quantization, dequantization, quantum optics, nuclear structure, and signal processing [39, 49, 53, 8, 10, 11, 12]. The Jacobi group was investigated by physicists under other names as *Hagen* [30], *Schrödinger* [47], or *Weyl-symplectic* group [62]. The Jacobi group is responsible for the squeezed states [38, 56, 43, 69, 33] in quantum optics [23, 44, 2, 55, 25].

The Jacobi group has been studied in the papers [8, 10] in connection with the group-theoretic approach to coherent states [48]. We have attached to the Jacobi group $G_n^J = H_n \rtimes \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ coherent states based on Siegel-Jacobi ball \mathcal{D}_n^J [10], which, as set, consists of the points of $\mathbb{C}^n \times \mathcal{D}_n$. H_n denotes the $(2n+1)$ -dimensional Heisenberg group and \mathcal{D}_n denotes the Siegel ball. The case G_1^J was studied in [8]. We have determined the Kähler two-form ω_n on \mathcal{D}_n^J from the Kähler potential [10] – the logarithm of the scalar product of two coherent states [7] – and, via the partial Cayley transform, we have determined the Kähler two-form ω'_n on the Siegel-Jacobi upper-half plane $\mathcal{X}_n^J = \mathbb{C}^n \times \mathcal{X}_n$, where \mathcal{X}_n is the Siegel upper half plane [9]. The Kähler two-form ω'_1 was investigated by Berndt [18] and Kähler [36, 37], while ω_n and ω'_n have been investigated also by Yang [65] - [68]. ω_n is written compactly as the sum of two terms, one describing the Kähler two-form on \mathcal{D}_n , $\omega_{\mathcal{D}_n}$, the other one is $\mathrm{Tr}(A^t(\mathbb{I}_n - \bar{W}W)^{-1} \wedge \bar{A})$, where $A = dz + dW\bar{\eta}$, and $\eta = (\mathbb{I}_n - W\bar{W})^{-1}(z + W\bar{z})$, $z \in \mathbb{C}^n$, $W \in \mathcal{D}_n$ [9, 10]. Let us denote by FC the change of variables $FC : \mathbb{C}^n \times \mathcal{D}_n \ni (\eta, W) \rightarrow (z, W) \in \mathcal{D}_n^J$, $z = \eta - W\bar{\eta}$. We put this change of variables in connection with the celebrated fundamental conjecture [60, 24] on the Siegel-Jacobi ball \mathcal{D}_n^J , as we did in [13, 14] for the Siegel-Jacobi disk \mathcal{D}_1^J . We also make similar considerations for the Siegel-Jacobi upper half plane \mathcal{X}_n^J .

In [4, 5] we have considered the problem of *dequantization* of a dynamical system with Lie group of symmetry G on a Hilbert space \mathfrak{H} in Berezin's approach [15, 17] in the simple case of linear Hamiltonians. Linear Hamiltonians in generators of the Jacobi group appear in many physical problems of quantum mechanics, as in the case of the quantum oscillator acted on by a variable external force [27, 51, 35].

What we find out is that the classical motion and quantum evolution on \mathcal{D}_n^J determined by linear Hamiltonians in the generators of the Jacobi group G_n^J are described by a matrix Riccati equation on \mathcal{D}_n and a first order coupled linear differential equation for $z \in \mathbb{C}^n$. The nice thing is that via the FC transform, the differential equation for $\dot{\eta}$ does not depend on $W \in \mathcal{D}_n$. The variables (η, W) appear to offer a convenient parametrization of the Siegel-Jacobi ball. Similar considerations are presented for the equations of motion on \mathcal{X}_n^J .

In the present paper we extend to G_n^J our results established in [14] for G_1^J .

The paper is laid out as follows. The notation for the Jacobi algebra \mathfrak{g}_n^J is fixed in §2. Starting with some notation on coherent states [48, 7], §3 deals with coherent states based on \mathcal{D}_n^J [10]. The holomorphic representation [6, 7] of the generators of the Jacobi group as first order differential operators with polynomial coefficients defined on \mathcal{D}_n^J is given in §4. It is verified that the differential realization of the generators of $\mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$ has the right hermiticity properties with respect to the scalar product of functions on \mathcal{D}_n - Lemma 2, and similarly for the generators of \mathfrak{g}_n^J - Lemma 3. In §5 we recall the expression of ω'_n and in Proposition 3 we show that the partial Cayley transform -the transform which connects \mathcal{D}_n^J and \mathcal{X}_n^J - is a Kähler homogenous diffeomorphism. In Proposition 4 of §6 we determine the FC -transform for the Siegel-Jacobi domains \mathcal{D}_n^J and \mathcal{X}_n^J . The proof is inspired by the paper [36] of Kähler. Corollary 1 expresses the reproducing kernel and the scalar product in the variables $(\eta, W) \in \mathbb{C}^n \times \mathcal{D}_n$. §7 is devoted to classical motion and quantum evolution on the Siegel-Jacobi domains determined by hermitian linear Hamiltonians in the generators of the Jacobi group G_n^J . The equations of motion are written down explicitly in Proposition 6 in §7.1 and their integration is discussed in §7.2. Use is made of the methods of [42] to integrate the matrix Riccati differential equation on manifolds by linearization, previously applied in [5] in the case of hermitian symmetric spaces. The last two paragraphs in §8 refer to the Berry phase [52] on \mathcal{D}_n^J and the energy function associated to the Hamiltonian linear in the generators of the Jacobi group G_n^J expressed in the variables $(\eta, W) \in \mathbb{C}^n \times \mathcal{D}_n$. In these variables the energy function is written down as the sum of a real function in η and one in W .

Notation. We denote by \mathbb{R} , \mathbb{C} , \mathbb{Z} , and \mathbb{N} the field of real numbers, the field of complex numbers, the ring of integers, and the set of non-negative integers, respectively. $M_{mn}(\mathbb{F}) \cong \mathbb{F}^{mn}$ denotes the set of all $m \times n$ matrices with entries in the field \mathbb{F} . $M_{n1}(\mathbb{F})$ is identified with \mathbb{F}^n . Set $M(n, \mathbb{F}) = M_{nn}(\mathbb{F})$. For any $A \in M(n, \mathbb{F})$, A^t denotes the transpose matrix of A , $A^s = (A + A^t)/2$ and $A^a = (A - A^t)/2$. For $A \in M_n(\mathbb{C})$, \bar{A} denotes the conjugate matrix of A and $A^* = \bar{A}^t$. For $A \in M_n(\mathbb{C})$, the inequality $A > 0$ means that A is positive definite. The identity matrix of degree n is denoted by \mathbb{I}_n and \mathbb{O}_n denotes the $M_n(\mathbb{F})$ -matrix with all entries 0. We denote by $\text{diag}(\alpha_1, \dots, \alpha_n)$ the matrix which has the elements $\alpha_1, \dots, \alpha_n$ on the diagonal and all the other elements 0. If A is a linear operator, we denote by A^\dagger its adjoint. We consider a complex separable Hilbert space \mathfrak{H} endowed with a scalar product which is antilinear in the first argument, $\langle \lambda x, y \rangle = \bar{\lambda} \langle x, y \rangle$, $x, y \in \mathfrak{H}$, $\lambda \in \mathbb{C} \setminus 0$. A complex analytic manifold is *Kählerian* if it is endowed with a Hermitian metric whose imaginary part ω has $d\omega = 0$ [32]. A coset space $M = G/H$ is *homogenous Kählerian* if it carries a Kählerian structure invariant under the group G [21]. By a *Kähler homogeneous diffeomorphism* we mean a diffeomorphism $\phi : M \rightarrow N$ of homogeneous Kähler manifolds such that $\phi^* \omega_N = \omega_M$. $\text{Ham}(M)$ denotes the Hamiltonian vector fields on the manifold M . We use Einstein convention that repeated indices are implicitly summed over. If $W = (w)_{ij}$ is a symmetric matrix, we introduce the symbols $\nabla_{ij} = \nabla_{ji} = \chi_{ij} \frac{\partial}{\partial w_{ij}}$, where $\chi_{ij} = \frac{1+\delta_{ij}}{2}$. In the expression $\chi_{ij} \frac{\partial}{\partial w_{ij}}$ of ∇_{ij} no summation is assumed.

2. THE JACOBI ALGEBRA \mathfrak{g}_n^J

Let \mathfrak{h}_n denotes the $(2n+1)$ -dimensional Heisenberg algebra, isomorphic to the algebra

$$(2.1) \quad \mathfrak{h}_n = \langle \text{is}1 + \sum_{i=1}^n (x_i a_i^\dagger - \bar{x}_i a_i) \rangle_{s \in \mathbb{R}, x_i \in \mathbb{C}},$$

where a_i^\dagger (a_i) are the boson creation (respectively, annihilation) operators, which verify the canonical commutation relations

$$(2.2) \quad [a_i, a_j^\dagger] = \delta_{ij}; \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0.$$

The displacement operator

$$(2.3) \quad D(\alpha) := \exp(\alpha a^\dagger - \bar{\alpha} a) = \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha a^\dagger) \exp(-\bar{\alpha} a),$$

verifies the composition rule:

$$(2.4) \quad D(\alpha_2)D(\alpha_1) = e^{i\theta_h(\alpha_2, \alpha_1)} D(\alpha_2 + \alpha_1), \quad \theta_h(\alpha_2, \alpha_1) := \Im(\alpha_2 \bar{\alpha}_1).$$

Here we have used the notation $\alpha\beta = \alpha_i\beta_i$, where $\alpha = (\alpha_i)_{i=1, \dots, n} \in \mathbb{C}^n$.

The composition law of the Heisenberg group H_n is:

$$(2.5) \quad (\alpha_2, t_2) \circ (\alpha_1, t_1) = (\alpha_2 + \alpha_1, t_2 + t_1 + \Im(\alpha_2 \bar{\alpha}_1)).$$

If we identify \mathbb{R}^{2n} with \mathbb{C}^n , $(p, q) \mapsto \alpha$:

$$(2.6) \quad \alpha = p + iq, \quad p, q \in \mathbb{R}^n,$$

then

$$\Im(\alpha_2 \bar{\alpha}_1) = (p_1^t, q_1^t) J \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}, \quad \text{where } J = \begin{pmatrix} \mathbb{O}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbb{O}_n \end{pmatrix}.$$

The Jacobi algebra is the the semi-direct sum $\mathfrak{g}_n^J := \mathfrak{h}_n \rtimes \mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$, where \mathfrak{h}_n is an ideal in \mathfrak{g}_n^J , i.e. $[\mathfrak{h}_n, \mathfrak{g}_n^J] = \mathfrak{h}_n$, determined by the commutation relations:

$$(2.7a) \quad [a_k^\dagger, K_{ij}^+] = [a_k, K_{ij}^-] = 0,$$

$$(2.7b) \quad [a_i, K_{kj}^+] = \frac{1}{2}\delta_{ik}a_j^\dagger + \frac{1}{2}\delta_{ij}a_k^\dagger, \quad [K_{kj}^-, a_i^\dagger] = \frac{1}{2}\delta_{ik}a_j + \frac{1}{2}\delta_{ij}a_k,$$

$$(2.7c) \quad [K_{ij}^0, a_k^\dagger] = \frac{1}{2}\delta_{jk}a_i^\dagger, \quad [a_k, K_{ij}^0] = \frac{1}{2}\delta_{ik}a_j.$$

The generators $K^{0,+, -}$ of $\mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$ verify the commutation relations

$$(2.8a) \quad [K_{ij}^-, K_{kl}^-] = [K_{ij}^+, K_{kl}^+] = 0, \quad 2[K_{ij}^-, K_{kl}^0] = K_{il}^- \delta_{kj} + K_{jl}^- \delta_{ki},$$

$$(2.8b) \quad 2[K_{ij}^-, K_{kl}^+] = K_{kj}^0 \delta_{li} + K_{lj}^0 \delta_{ki} + K_{ki}^0 \delta_{lj} + K_{li}^0 \delta_{kj},$$

$$(2.8c) \quad 2[K_{ij}^+, K_{kl}^0] = -K_{ik}^+ \delta_{jl} - K_{jk}^+ \delta_{li}, \quad 2[K_{ji}^0, K_{kl}^0] = K_{jl}^0 \delta_{ki} - K_{ki}^0 \delta_{lj}.$$

Now we briefly fix the notation concerning the symplectic group. For $A \in \text{GL}(2n, \mathbb{F})$, we have (here \mathbb{F} is any of the fields \mathbb{R}, \mathbb{C}): $A \in \text{Sp}(n, \mathbb{F}) \Leftrightarrow A^t J A = J$. Consequently, a matrix $X \in \mathfrak{gl}(2n, \mathbb{F})$ is in $\mathfrak{sp}(n, \mathbb{F})$ iff $X^t J + J X = 0$. The matrices from $\mathfrak{sp}(n, \mathbb{R})$ are also called *infinitesimally symplectic* or *Hamiltonians* [46].

We recall also that $g \in \text{U}(n, n)$ iff $gKg^* = K$, where $K = \begin{pmatrix} \mathbb{I}_n & \mathbb{O}_n \\ \mathbb{O}_n & -\mathbb{I}_n \end{pmatrix}$.

We summarize some properties of symplectic and Hamiltonian matrices (cf. [54, 3, 28, 46]; for the characterization of the eigenvalues, see [46, 1, 40, 63]):

Remark 1. *a) X is a Hamiltonian matrix iff one of the following equivalent conditions are fulfilled:*

- 1) $X^t J + JX = 0$;
- 2) $X = JR$, where $R \in M(2n, \mathbb{R})$ is a symmetric matrix;
- 3) JX is symmetric;
- 4) $X \in M(2n, \mathbb{R})$ has the form

$$(2.9) \quad X = \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} \in \mathfrak{sp}(n, \mathbb{R}), \quad b = b^t, \quad c = c^t, \quad a, b, c \in M(n, \mathbb{R});$$

b) If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(2n, \mathbb{R})$, then the matrices $a, b, c, d \in M(n, \mathbb{R})$ have the properties

$$(2.10a) \quad a^t c = c^t a, \quad b^t d = d^t b, \quad a^t d - c^t b = \mathbb{I}_n;$$

$$(2.10b) \quad ab^t = ba^t, \quad cd^t = dc^t, \quad ad^t - bc^t = \mathbb{I}_n.$$

c) Under the identification (2.6) of \mathbb{R}^{2n} with \mathbb{C}^n , we have the correspondence

$$(2.11) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2n, \mathbb{R}) \leftrightarrow M_{\mathbb{C}} = \mathcal{C}^{-1} M \mathcal{C} = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}, \quad p, q \in M(n, \mathbb{C}),$$

where

$$(2.12) \quad \mathcal{C} = \begin{pmatrix} i\mathbb{I}_n & i\mathbb{I}_n \\ -\mathbb{I}_n & \mathbb{I}_n \end{pmatrix}, \quad \mathcal{C}^{-1} = \frac{1}{2} \begin{pmatrix} -i\mathbb{I}_n & -\mathbb{I}_n \\ -i\mathbb{I}_n & \mathbb{I}_n \end{pmatrix};$$

$$(2.13a) \quad 2a = p + q + \bar{p} + \bar{q}, \quad 2b = i(\bar{p} - \bar{q} - p + q),$$

$$(2.13b) \quad 2c = i(p + q - \bar{p} - \bar{q}), \quad 2d = p - q + \bar{p} - \bar{q};$$

$$(2.14) \quad 2p = a + d + i(b - c), \quad 2q = a - d - i(b + c).$$

In particular, to the Hamiltonian matrix (2.9) X we associate $X_{\mathbb{C}} = \mathcal{C}^{-1} X \mathcal{C} \in \mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}} = \mathfrak{sp}(n, \mathbb{C}) \cap \mathfrak{u}(n, n)$,

$$(2.15) \quad X_{\mathbb{C}} = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}, \quad p^* = -p, \quad q^t = q,$$

where

$$(2.16) \quad 2p = a - a^t + i(b - c); \quad 2q = a + a^t - i(b + c).$$

The relations inverse to (2.16) are

$$(2.17) \quad 2a = p + \bar{p} + q + \bar{q}; \quad 2b = i(\bar{p} - p + q - \bar{q}); \quad 2c = i(p + q - \bar{p} - \bar{q}).$$

Also, we have

$$(2.18) \quad X \in \mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}} \quad \text{iff} \quad X = iKH, \quad H^* = H, \quad H \in M(2n, \mathbb{C}).$$

d) To every $g \in \mathrm{Sp}(n, \mathbb{R})$, we associate via (2.11) $g \mapsto g_{\mathbb{C}} \in \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}} \equiv \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{U}(n, n)$

$$(2.19) \quad g_{\mathbb{C}} = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix},$$

where the matrices $p, q \in M(n, \mathbb{C})$ have the properties

$$(2.20a) \quad pp^* - qq^* = \mathbb{I}_n, \quad pq^t = qp^t;$$

$$(2.20b) \quad p^*p - q^t\bar{q} = \mathbb{I}_n, \quad p^t\bar{q} = q^*p.$$

e) The characteristic polynomial of a real Hamiltonian matrix is an even polynomial. If λ is an eigenvalue of a Hamiltonian matrix with multiplicity k , so are $-\lambda, \bar{\lambda}, -\bar{\lambda}$ with the same multiplicity. Moreover, 0, if it occurs, has even multiplicity. If $A \in \mathfrak{sp}(n, \mathbb{R})$ has distinct eigenvalues $\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n$, there exists a symplectic matrix S (possibly complex) such as $S^{-1}AS = \mathrm{diag}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n)$.

The characteristic polynomial of a symplectic matrix is a reciprocal polynomial. If λ is an eigenvalue of a real symplectic matrix with multiplicity k , so are $\lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$ with the same multiplicity. Moreover, the multiplicities of the eigenvalues $+1$ and -1 , if they occur, are even.

3. COHERENT STATES ON \mathcal{D}_n^J

In order to fix the notation on coherent states [48], let us consider the triplet (G, π, \mathfrak{H}) , where π is a continuous, unitary representation of the Lie group G on the separable complex Hilbert space \mathfrak{H} .

For $X \in \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of the Lie group G , let us define the (unbounded) operator $d\pi(X)$ on \mathfrak{H} by $d\pi(X).v := d/dt|_{t=0} \pi(\exp tX).v$, whenever the limit on the right hand side exists. We obtain a representation of the Lie algebra \mathfrak{g} on \mathfrak{H}^∞ (\mathfrak{H}^∞ denotes the smooth vectors of \mathfrak{H}), the derived representation, and we denote $\mathbf{X}.v := d\pi(X).v$ for $X \in \mathfrak{g}, v \in \mathfrak{H}^\infty$.

Let us now denote by H the isotropy group. We consider (generalized) coherent states on complex homogeneous manifolds $M \cong G/H$ [48]. The coherent vector mapping is defined locally, on a coordinate neighborhood \mathcal{V}_0 , $\varphi : M \rightarrow \bar{\mathfrak{H}}$, $\varphi(z) = e_{\bar{z}}$ (cf. [6, 7]), where $\bar{\mathfrak{H}}$ denotes the Hilbert space conjugate to \mathfrak{H} . The vectors $e_{\bar{z}} \in \bar{\mathfrak{H}}$ indexed by the points $z \in M$ are called *Perelomov's coherent state vectors*. Explicitly, $e_z = \exp(\sum_{\alpha \in \Delta_+} z_\alpha \mathbf{X}_\alpha) e_0$, where e_0 is the extremal weight vector of the representation π , Δ_+ are the positive roots of the Lie algebra \mathfrak{g} of G , and $X_\alpha, \alpha \in \Delta$, are the generators [48].

The space $\mathcal{F}_{\bar{\mathfrak{H}}}$ of holomorphic functions is defined as the set of square integrable functions with respect to the scalar product

$$(3.1) \quad (f, g)_{\mathcal{F}_{\bar{\mathfrak{H}}}} = \int_M \bar{f}(z)g(z) d\nu_M(z, \bar{z}), \quad d\nu_M(z, \bar{z}) = \frac{\Omega_M(z, \bar{z})}{(e_{\bar{z}}, e_{\bar{z}})}.$$

Here Ω_M is the normalized G -invariant volume form

$$(3.2) \quad \Omega_M := (-1)^{\binom{n}{2}} \frac{1}{n!} \underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ times}},$$

and the G -invariant Kähler two-form ω on the $2n$ -dimensional manifold M is given by

$$(3.3) \quad \omega(z) = i \sum_{\alpha \in \Delta_+} \mathcal{G}_{\alpha, \beta} d z_\alpha \wedge d \bar{z}_\beta,$$

$$(3.4) \quad \mathcal{G}_{\alpha, \beta} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log \langle e_{\bar{z}}, e_{\bar{z}} \rangle, \quad \mathcal{G}_{\alpha, \beta} = \bar{\mathcal{G}}_{\beta, \alpha}.$$

(3.1) is nothing else but *Parseval overcompleteness identity* [16].

Let us now introduce the map $\Phi : \mathfrak{H}^* \rightarrow \mathcal{F}_{\mathfrak{H}}$,

$$(3.5) \quad \Phi(\psi) := f_\psi, f_\psi(z) = \Phi(\psi)(z) = (\varphi(z), \psi)_{\mathfrak{H}} = (e_{\bar{z}}, \psi)_{\mathfrak{H}}, \quad z \in \mathcal{V}_0, \quad \mathcal{V}_0 \subset M,$$

where we have identified the space $\bar{\mathfrak{H}}$ complex conjugate to \mathfrak{H} with the dual space \mathfrak{H}^* of \mathfrak{H} .

Perelomov's coherent state vectors [48] associated to the group G_n^J with Lie algebra the Jacobi algebra \mathfrak{g}_n^J , based on the complex N -dimensional ($N = \frac{n(n+3)}{2}$) manifold - the Siegel-Jacobi ball $\mathcal{D}_n^J := H_n/\mathbb{R} \times \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}/\mathrm{U}(n) = \mathbb{C}^n \times \mathcal{D}_n$ - are defined as [10, 48]

$$(3.6) \quad e_{z, W} = \exp(\mathbf{X})e_0, \quad \mathbf{X} := \sum_{i=1}^n z_i \mathbf{a}_i^\dagger + \sum_{i, j=1}^n w_{ij} \mathbf{K}_{ij}^+, \quad z \in \mathbb{C}^n; W \in \mathcal{D}_n.$$

The non-compact hermitian symmetric space $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}/\mathrm{U}(n)$ admits a matrix realization as a bounded homogeneous domain, the Siegel ball \mathcal{D}_n

$$(3.7) \quad \mathcal{D}_n := \{W \in M(n, \mathbb{C}) : W = W^t, \mathbb{I}_n - W\bar{W} > 0\}.$$

\mathcal{D}_n is a hermitian symmetric space of type CI (cf. Table V, p. 518, in [32]), identified with the symmetric bounded domain of type II, \mathfrak{R}_{II} in Hua's notation [34].

The vector e_0 appearing in (3.6) verifies the relations

$$(3.8) \quad \mathbf{a}_i e_0 = 0, \quad i = 1, \dots, n; \quad \mathbf{K}_{ij}^+ e_0 \neq 0, \quad \mathbf{K}_{ij}^- e_0 = 0, \quad \mathbf{K}_{ij}^0 e_0 = \frac{k_i}{4} \delta_{ij} e_0, \quad i, j = 1, \dots, n.$$

In (3.8), $e_0 = e_0^H \otimes e_0^K$, where e_0^H is the minimum weight vector (vacuum) for the Heisenberg group H_n with respect to the representation (2.3), while e_0^K is the extremal weight vector for $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ corresponding to the weight k in (3.8) with respect to a unitary representation S , see details in [10].

The following proposition describes the holomorphic action of the Jacobi group on the Siegel-Jacobi ball and some geometric properties of \mathcal{D}_n^J (cf. [10]):

Proposition 1. *Let us consider the action $S(g)D(\alpha)e_{z, W}$, where $g \in \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ is of the form (2.19), (2.20), $D(\alpha)$ is given by (2.3) and the coherent state vector is defined in (3.6). Then formulas (3.9) hold:*

$$(3.9a) \quad S(g)D(\alpha)e_{z, W} = \lambda e_{z_1, W_1}, \quad \lambda = \lambda(g, \alpha; z, W),$$

$$(3.9b) \quad W_1 = g \cdot W = (pW + q)(\bar{q}W + \bar{p})^{-1} = (Wq^* + p^*)^{-1}(q^t + Wp^t),$$

$$(3.9c) \quad z_1 = (Wq^* + p^*)^{-1}(z + \alpha - W\bar{\alpha}),$$

$$(3.9d) \quad \lambda = \det(Wq^* + p^*)^{-k/2} \exp\left(\frac{\bar{\eta}}{2}z - \frac{\bar{\eta}}{2}z_1\right) \exp(i\theta_h(\alpha, \eta)),$$

$$(3.9e) \quad \eta = (\mathbb{I}_n - W\bar{W})^{-1}(z + W\bar{z}),$$

$$(3.9f) \quad \eta_1 = p(\alpha + \eta) + q(\bar{\alpha} + \bar{\eta}).$$

The action of the Jacobi group G_n^J on the manifold \mathcal{D}_n^J is given by equations (3.9c), (3.9b). The composition law is

$$(3.10) \quad (g_1, \alpha_1, t_1) \circ (g_2, \alpha_2, t_2) = (g_1 \circ g_2, g_2^{-1} \cdot \alpha_1 + \alpha_2, t_1 + t_2 + \Im(g_2^{-1} \cdot \alpha_1 \bar{\alpha}_2)),$$

and if g is as in (2.19), then $g \cdot \alpha := \alpha_g$ is given by $\alpha_g = p\alpha + q\bar{\alpha}$, and $g^{-1} \cdot \alpha = p^*\alpha - q^t\bar{\alpha}$.

The manifold \mathcal{D}_n^J has the Kähler potential (3.11), $f = \log K$, with K given by (3.14),

$$(3.11) \quad f = -\frac{k}{2} \log \det(\mathbb{I}_n - W\bar{W}) + \bar{z}^t(\mathbb{I}_n - W\bar{W})^{-1}z \\ + \frac{1}{2}z^t[\bar{W}(\mathbb{I}_n - W\bar{W})^{-1}]z + \frac{1}{2}\bar{z}^t[(\mathbb{I}_n - W\bar{W})W]\bar{z}.$$

The Kähler two-form ω_n , deduced as in (3.3), G_n^J -invariant to the action (3.9b), (3.9c) is

$$(3.12) \quad -i\omega_n = \frac{k}{2} \text{Tr}(B \wedge \bar{B}) + \text{Tr}(A^t \bar{M} \wedge \bar{A}), \quad A = dz + dW\bar{\eta}, \\ B = M dW, \quad M = (\mathbb{I}_n - W\bar{W})^{-1}, \quad \eta = M(z + W\bar{z}).$$

The scalar product $K : M \times \bar{M} \rightarrow \mathbb{C}$, $K(\bar{x}, \bar{V}; y, W) = (e_{x,V}, e_{y,W})_k$ is:

$$(3.13) \quad (e_{x,V}, e_{y,W})_k = \det(U)^{k/2} \exp F(\bar{x}, \bar{V}; y, W), \quad U = (\mathbb{I}_n - W\bar{V})^{-1}; \\ 2F(\bar{x}, \bar{V}; y, W) = 2\langle x, Uy \rangle + \langle V\bar{y}, Uy \rangle + \langle x, UW\bar{x} \rangle.$$

In particular, the reproducing kernel $K = (e_{z,W}, e_{z,W})$ is

$$(3.14) \quad K = \det(M)^{\frac{k}{2}} \exp F, \quad M = (\mathbb{I}_n - W\bar{W})^{-1},$$

$$(3.15) \quad 2F = 2\bar{z}^t M z + z^t \bar{W} M z + \bar{z}^t M W \bar{z}.$$

The Hilbert space of holomorphic functions \mathcal{F}_K associated to the holomorphic kernel K given by (3.14) is endowed with the scalar product of the type (3.1)

$$(3.16) \quad (\phi, \psi) = \Lambda_n \int_{z \in \mathbb{C}^n; \mathbb{I}_n - W\bar{W} > 0} \bar{f}_\phi(z, W) f_\psi(z, W) Q K^{-1} dz dW,$$

where the normalization constant Λ_n is given by (3.17)

$$(3.17) \quad \Lambda_n = \frac{k-3}{2\pi^{n(n+3)/2}} \prod_{i=1}^{n-1} \frac{(\frac{k-3}{2} - n + i) \Gamma(k+i-2)}{\Gamma[k+2(i-n-1)]}.$$

and the density of volume given is given by (3.18)

$$(3.18) \quad Q = \det(\mathbb{I}_n - W\bar{W})^{-(n+2)}, \quad dz = \prod_{i=1}^n d\Re z_i d\Im z_i; \quad dW = \prod_{1 \leq i \leq j \leq n} d\Re w_{ij} d\Im w_{ij}.$$

Comparatively with the case of the symplectic group, a shift of p to $p - 1/2$ in the normalization constant (4.12) $\Lambda_n = \pi^{-n} J^{-1}(p)$ is obtained. We write down the scalar product (3.16) also as ($p = k/2 - n - 2$)

$$(3.19) \quad (\phi, \psi) = \Lambda_n \int_{z \in \mathbb{C}^n; W \in \mathcal{D}_n} \bar{f}_\phi(z, W) f_\psi(z, W) \rho_1 dz dW, \quad \rho_1 = \det(\mathbb{I}_n - W\bar{W})^p \exp -F.$$

4. THE DIFFERENTIAL ACTION

Let us consider again the triplet (G, π, \mathfrak{H}) introduced at the beginning of §3. The unitarity and the continuity of the representation π imply that $\text{id } \pi(X)|_{\mathfrak{H}^\infty}$ is essentially selfadjoint. Let us denote his image in $B_0(\mathfrak{H}^\infty)$ by $\mathbf{A}_M := \text{d } \pi(\mathcal{U}(\mathfrak{g}_\mathbb{C}))$, where B_0 denotes the set of linear operators with formal adjoint, and $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ denotes the universal covering algebra. If $\Phi : \mathfrak{H}^* \rightarrow \mathcal{F}_\mathfrak{H}$ is the isometry (3.5), we are interested in the study of the image of \mathbf{A}_M via Φ as subset in the algebra of holomorphic, linear differential operators, $\Phi \mathbf{A}_M \Phi^{-1} := \mathbb{A}_M \subset \mathfrak{D}_M$.

The set \mathfrak{D}_M (or simply \mathfrak{D}) of *holomorphic, finite order, linear differential operators on M* is a subalgebra of homomorphisms $\mathcal{H}om_\mathbb{C}(\mathcal{O}_M, \mathcal{O}_M)$ generated by the set \mathcal{O}_M of germs of holomorphic functions of M and the vector fields. We consider also the subalgebra \mathfrak{A}_M of \mathbb{A}_M of *differential operators with holomorphic polynomial coefficients*. Let $U := \mathcal{V}_0 \subset M$, endowed with the local coordinates (z_1, z_2, \dots, z_n) . We set $\partial_i := \frac{\partial}{\partial z_i}$ and $\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$, $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$. The sections of \mathfrak{D}_M on U are $A : f \mapsto \sum_\alpha a_\alpha \partial^\alpha f$, $a_\alpha \in \Gamma(U, \mathcal{O})$, a_α -s being zero except a finite number.

For $k \in \mathbb{N}$, let us denote by \mathfrak{D}_k the subset of differential operators of degree $\leq k$. The filtration of \mathfrak{D} induces a filtration on \mathfrak{A} .

Summarizing, we have a correspondence between the following three objects [6, 7]:

$$(4.1) \quad \mathfrak{g}_\mathbb{C} \ni X \mapsto \mathbf{X} \in \mathbf{A}_M \mapsto \mathbb{X} \in \mathbb{A}_M \subset \mathfrak{D}_M, \text{ differential operator on } \mathcal{F}_\mathfrak{H}.$$

Moreover, it is easy to see [6] that *if Φ is the isometry (3.5), then $\Phi d\pi(\mathfrak{g}_\mathbb{C})\Phi^{-1} \subseteq \mathfrak{D}_1$ and we have*

$$(4.2) \quad \mathfrak{g}_\mathbb{C} \ni X \mapsto \mathbb{X} \in \mathfrak{D}_1; \quad \mathbb{X}_z(f_\psi(z)) = \mathbb{X}_z(e_{\bar{z}}, \psi) = (e_{\bar{z}}, \mathbf{X}\psi),$$

where

$$(4.3) \quad \mathbb{X}_z(f_\psi(z)) = \left(P_X(z) + \sum Q_X^i(z) \frac{\partial}{\partial z_i} \right) f_\psi(z).$$

In [6, 7] we have advanced the hypothesis that for *coherent state groups the holomorphic functions P and Q in (4.3) are polynomials*, i.e. $\mathbb{A} \subset \mathfrak{A}_1 \subset \mathfrak{D}_1$.

In particular, the Jacobi algebra \mathfrak{g}_n^J admits a realization in the space \mathfrak{D}_1 of holomorphic first order differential operators with polynomial coefficients, defined on the Siegel-Jacobi ball \mathcal{D}_n^J . The space of holomorphic functions on which the differential operators act is the space denoted \mathcal{F}_K in Proposition 1. For explicit realization of the representation [4, 5, 7], use is made of the formula $\text{Ad}(\exp X) = \exp(\text{ad}_X)$, i.e.

$$(4.4) \quad A e^X = e^X (A - [X, A] + \frac{1}{2} [X, [X, A]] + \dots),$$

In order to take into account the symmetry of the matrix W appearing in (3.6), we use the derivation formula:

$$(4.5) \quad \frac{\partial w_{ij}}{\partial w_{pq}} = \delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp} - \delta_{ij} \delta_{pq} \delta_{ip}, \quad w_{ij} = w_{ji}.$$

With (4.4) and taking into account the commutation relations (2.2), (2.7) and (2.8) of the generators of the Jacobi algebra, we get the relations (4.6) (see also §2.4, 3.3 in

[10], where χ has been omitted, and [49]):

$$(4.6a) \quad \mathbf{a}_k^\dagger e_{z,W} = \frac{\partial}{\partial z_k} e_{z,W}, \quad \mathbf{a}_k e_{z,W} = \left(z_k + w_{ki} \frac{\partial}{\partial z_i} \right) e_{z,W};$$

$$(4.6b) \quad \mathbf{K}_{kl}^0 e_{z,W} = \left(\frac{k_k}{4} \delta_{kl} + \frac{z_l}{2} \frac{\partial}{\partial z_k} + w_{li} \nabla_{ik} \right) e_{z,W}, \quad \mathbf{K}_{kl}^+ e_{z,W} = \nabla_{kl} e_{z,W};$$

$$(4.6c) \quad \mathbf{K}_{kl}^- e_{z,W} = \left[\frac{k_k + k_l}{4} w_{kl} + \frac{z_k z_l}{2} + \frac{1}{2} (z_l w_{ik} + z_k w_{il}) \frac{\partial}{\partial z_i} + w_{\alpha l} w_{ki} \nabla_{i\alpha} \right] e_{z,W}.$$

Now we briefly recall some general considerations (for details see [6, 8] and Lemma 1 in [7]). For $X \in \mathfrak{g}$, let $\mathbf{X}.e_z := \mathbf{X}_z e_z$. Then $\mathbf{X}.e_{\bar{z}} = \mathbf{X}_{\bar{z}} e_{\bar{z}}$.

But $(e_{\bar{z}}, \mathbf{X}.e_{\bar{z}'}) = (\mathbf{X}^\dagger.e_{\bar{z}}, e_{\bar{z}'})$ and finally, with equation (4.2), we have

$$(4.7) \quad \mathbb{X}_{\bar{z}'}(e_{\bar{z}}, e_{\bar{z}'}) = \mathbb{X}_z^\dagger(e_{\bar{z}}, e_{\bar{z}'})$$

Using (4.6) and the relation expressed in (4.7), we have

Lemma 1. *The Jacobi algebra \mathfrak{g}_n^J admits a realization in the space \mathfrak{D}_1 of differential operators on \mathcal{D}_n^J :*

$$(4.8a) \quad \mathbf{a}_k = \frac{\partial}{\partial z_k}, \quad \mathbf{a}_k^\dagger = z_k + w_{ki} \frac{\partial}{\partial z_i};$$

$$(4.8b) \quad \mathbf{K}_{kl}^0 = \frac{k_k}{4} \delta_{kl} + \frac{z_k}{2} \frac{\partial}{\partial z_l} + w_{ki} \nabla_{il}, \quad \mathbf{K}_{kl}^- = \nabla_{kl};$$

$$(4.8c) \quad \mathbf{K}_{kl}^+ = \frac{k_k + k_l}{4} w_{kl} + \frac{z_k z_l}{2} + \frac{1}{2} (z_l w_{ik} + z_k w_{il}) \frac{\partial}{\partial z_i} + w_{\alpha l} w_{ki} \nabla_{i\alpha}.$$

In the formulae above, $k, l = 1, \dots, n, w_{kl} = w_{lk}$, $\frac{\partial}{\partial w_{kl}} = \frac{\partial}{\partial w_{lk}}$, and the dummy summation is on all indexes $1, \dots, n$.

With the convention $\nabla = (\nabla_{ij})_{i,j=1,\dots,n} = (\chi_{ij} \frac{\partial}{\partial w_{ij}})_{i,j=1,\dots,n}$, formulae (4.8) can be also written as

$$(4.9a) \quad \mathbf{a} = \frac{\partial}{\partial z}, \quad \mathbf{a}^\dagger = z + W \frac{\partial}{\partial z};$$

$$(4.9b) \quad \mathbb{K}^- = \nabla_W, \quad \mathbb{K}^0 = \frac{k}{4} + \frac{1}{2} \frac{\partial}{\partial z} \otimes z + \nabla_W W;$$

$$(4.9c) \quad \mathbb{K}^+ = \frac{W'}{4} + \frac{1}{2} z \otimes z + \frac{1}{2} (W \frac{\partial}{\partial z} \otimes z + z \otimes \frac{\partial}{\partial z} W) + W \nabla_W W.$$

In (4.9) $A \otimes B$ denotes the Kronecker product of matrices, here $(A \otimes B)_{kl} = a_k b_l$, $A = (a_k)$, $B = (b_l)$, $k = \text{diag}(k_1, \dots, k_n)$, $w'_{kl} = (k_k + k_l) w_{kl}$, $k, l = 1, \dots, n$.

We particularize the values of the operators given in Lemma 1 in the case of the action of $\text{Sp}(n, \mathbb{R})_{\mathbb{C}}$ on \mathcal{D}_n and we have

Lemma 2. *The algebra $\mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$ defined by the commutations relations (2.8) admits the realization in differential operators on the Siegel ball \mathcal{D}_n*

$$(4.10) \quad \mathbf{K}_{kl}^0 = \frac{k_k}{4} \delta_{kl} + w_{ki} \nabla_{il}, \quad \mathbf{K}_{kl}^- = \nabla_{kl}, \quad \mathbf{K}_{kl}^+ = \frac{k_k + k_l}{4} w_{kl} + w_{\alpha l} w_{ki} \nabla_{i\alpha}.$$

The generators of $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ have the hermiticity properties

$$(4.11) \quad (\mathbf{K}^+)_{kl}^\dagger = \mathbf{K}_{lk}^- = \mathbf{K}_{kl}^-, (\mathbf{K}^0)_{kl}^\dagger = \mathbf{K}_{lk}^0$$

with respect to the scalar product ([10])

$$(4.12) \quad (\phi, \psi)_{\mathcal{F}_S} = \Lambda'_n \int_{1-W\bar{W}>0} \bar{f}_\phi(W) f_\psi(W) \rho \, dW,$$

where $k_i = k$, $i = 1, \dots, n$ in (3.8),

$$(4.13) \quad \rho = \det(\mathbb{I}_n - W\bar{W})^q, \quad q = \frac{k}{2} - n - 1$$

and $\Lambda'_n = J_n^{-1}(q)$, with $J_n(p)$ given by (4.14), $p > -1$,

$$(4.14) \quad J_n(p) = 2^n \pi^{\frac{n(n+1)}{2}} \prod_{i=1}^n \frac{\Gamma(2p+2i)}{\Gamma(2p+n+i+1)} = \frac{\pi^{n(n+1/2)}}{p+n} \prod_{i=1}^{n-1} \frac{\Gamma[2(i+p-1)]}{(i+p)\Gamma[i+2(p+n-1)]}.$$

Proof. Firstly we have to check up that the operators (4.10) verify the commutation relations (2.8). This is a long calculation, based on the formula (4.5).

Then we verify the relation $(\mathbf{K}_{kl}^+ f, g)_{\mathcal{F}_S} = (f, \mathbf{K}_{lk}^- g)_{\mathcal{F}_S}$, which imposes to the reproducing kernel ρ the condition

$$(4.15) \quad (\nabla_{kl} + q\bar{w}_{kl} - \bar{w}_{ki}\bar{w}_{lj}\bar{\nabla}_{ij})\rho = 0.$$

The hermiticity condition $(\mathbf{K}^0)_{kl}^\dagger = \mathbf{K}_{lk}^0$ of the operators given by the first formula (4.10) with respect to the scalar product (4.12), with ρ given by (4.13), imposes to the kernel function ρ the condition:

$$(4.16) \quad (\bar{w}_{kp}\bar{\nabla}_{lp} - w_{lp}\nabla_{kp})\rho = 0.$$

The conditions (4.15) and (4.16) are verified by the kernel (4.13) using the relation

$$(4.17) \quad \frac{\partial}{\partial w_{ik}} A = (-2X_{ik} + X_{ik}\delta_{ik})A, \text{ or } \nabla A = -X, \text{ where } A = \det(\mathbb{I}_n - W\bar{W}),$$

which implies

$$(4.18) \quad \frac{\partial \rho}{\partial w_{ab}} = q(-2X_{ab} + X_{ab}\delta_{ab})\rho, \text{ or } \nabla \rho = -qX\rho, \text{ where } X = X^t = \bar{W}(\mathbb{I}_n - W\bar{W})^{-1}.$$

Indeed, with (4.18), the condition (4.15) reads $-X + \bar{W} + \bar{W}\bar{X}\bar{W} = 0$, while (4.16) reads $(\bar{X}\bar{W})^t = XW$. The last two conditions are verified because of the symmetry of the matrices X and W . \square

Lemma 3. The pairs of operators \mathbf{a}^\dagger and \mathbf{a} , \mathbf{K}_{kl}^+ and \mathbf{K}_{kl}^- , \mathbf{K}_{kl}^0 and \mathbf{K}_{lk}^0 are respectively hermitian conjugate with respect the scalar product (3.19) for $k_i = k$ in (3.8).

Proof. We take the derivative of (3.15) with respect with z_i and we find successively

$$\begin{aligned} \frac{\partial F}{\partial z_i} &= \bar{z}_p M_{pi} + \frac{1}{2}[(\bar{W}M)_{iq}z_q + z_p(\bar{W}M)_{pi}] \\ &= [M^t\bar{z} + \frac{1}{2}(\bar{W}Mz + \bar{M}\bar{W}z)]_i \\ &= [M^t(\bar{z} + \bar{W}z)]_i, \end{aligned}$$

and we introduce $\eta = M(z + W\bar{z})$, $M = (\mathbb{I}_n - W\bar{W})^{-1}$. We get

$$(4.19) \quad \frac{\partial F}{\partial z_i} = \bar{\eta}_i,$$

$$(4.20) \quad \frac{1}{\rho_1} \frac{\partial \rho_1}{\partial z_i} = -\bar{\eta}_i.$$

We look for the derivative of ρ_1 from (3.19) with respect to the w_{ik} , and we have

$$(4.21) \quad \frac{\partial \rho_1}{\partial w_{ik}} = (p \frac{\partial A}{\partial w_{ik}} - A \frac{\partial F}{\partial w_{ik}}) A^{p-1} \exp(-F), \text{ where } A = \det(\mathbb{I}_n - W\bar{W}),$$

and

$$(4.22) \quad \frac{\partial F}{\partial w_{ik}} = \bar{\eta}_i \bar{\eta}_k - \frac{1}{2} \bar{\eta}_i \bar{\eta}_k \delta_{ik}, \quad \text{or } \nabla F = \frac{1}{2} \bar{\eta} \otimes \bar{\eta}.$$

Indeed, with formula (4.5), we get

$$(4.23a) \quad \frac{\partial M_{ab}}{\partial w_{ik}} = M_{ai} X_{kb} + M_{ak} X_{ib} - M_{ai} X_{ib} \delta_{ik},$$

$$(4.23b) \quad \frac{\partial X_{ab}}{\partial w_{ik}} = X_{ai} X_{bk} + X_{ak} X_{ib} - X_{ai} X_{ib} \delta_{ik},$$

$$(4.23c) \quad \frac{\partial \bar{X}_{ab}}{\partial w_{ik}} = M_{ai} M_{bk} + M_{ak} M_{bi} - M_{ai} M_{bk} \delta_{ik}.$$

In order to prove (4.23c), we use the fact that $\bar{X} = MW$, then we use (4.23a), (4.5), and the formula $\mathbb{I}_n + XW = \bar{M} = M^t$.

We write (3.15) as

$$2F = 2\bar{z}^t M z + z^t X z + \bar{z}^t \bar{X} \bar{z}$$

and, with (4.23), we get

$$\begin{aligned} 2 \frac{\partial F}{\partial w_{ik}} = & 2[(\bar{z}^t M)_i (X z)_k + (\bar{z}^t M)_k (X z)_i - (\bar{z}^t M)_i (X z)_k \delta_{ik}] + \\ & + (z^t X)_i (X z)_k + (z^t X)_k (X z)_i - (z^t X)_i (X z)_k \delta_{ik} + \\ & + (\bar{z}^t M)_i (\bar{z}^t M)_k + (\bar{z}^t M)_k (\bar{z}^t M)_i - (\bar{z}^t M)_i (\bar{z}^t M)_k \delta_{ik}. \end{aligned}$$

Then we use twice the relations $(\bar{z}^t M + z^t X)^t = \bar{\eta}$, and (4.22) get proved.

With (4.17) and (4.22), we have for (4.21) the expression

$$(4.24) \quad -\frac{1}{\rho_1} \frac{\partial \rho_1}{\partial w_{ik}} = p(2X_{ik} - X_{ik} \delta_{ik}) + \bar{\eta}_i \bar{\eta}_k - \frac{1}{2} \bar{\eta}_i \bar{\eta}_k \delta_{ik}, \quad \text{and } -\frac{1}{\rho_1} \nabla \rho_1 = pX + \frac{1}{2} \bar{\eta} \otimes \bar{\eta}.$$

We have to verify $(\mathbf{a}_k f, g) = (f, \mathbf{a}_k^\dagger g)$ with respect to the scalar product (3.19), i.e.

$$\frac{\partial F}{\partial \bar{z}_k} = z_k + w_{ki} \frac{\partial F}{\partial z_i}.$$

With (4.20), the last condition reads $\eta = z + W\bar{\eta}$, which is true.

In order to verify $(\mathbf{K}_{kl}^0 f, g) = (f, \mathbf{K}_{lk}^0 g)$ for the case $k_i = k$ in formula (3.8) with respect to the scalar product (3.19), we use the differential action for \mathbf{K}_{kl}^0 in Lemma 1.

If we denote the integrant in the second term by f_{kl} , using (4.22), (4.24), (4.5) and the formula $z = \eta - W\bar{\eta}$, we find

$$\frac{f_{kl}}{\rho_1} = \frac{p}{2}\delta_{kl} + \frac{1}{2}\eta_l\bar{\eta}_k + p(W\bar{M}\bar{W})_{lk},$$

and $f_{kl} = \bar{f}_{lk}$, because the symmetry of the matrices W and X .

We also find for the integrant of $(\mathbf{K}_{kl}^- f, g) = (f, \mathbf{K}_{kl}^+ g)$ the common value $[p\bar{X}_{kl} + \frac{1}{2}\eta_l\eta_k]\rho_1$. \square

5. THE REAL JACOBI GROUP $G_n^J(\mathbb{R})$

We consider the real Jacobi group $G_n^J(\mathbb{R}) = \text{Sp}(n, \mathbb{R}) \ltimes H_n$, where H_n is now the real Heisenberg group of real dimension $(2n + 1)$. Let $g = (M, X, k), g' = (M', X', k') \in G_n^J(\mathbb{R})$, where $X = (\lambda, \mu) \in \mathbb{R}^{2n}$ and $(X, k) \in H_n$. Then the composition law in $G_n^J(\mathbb{R})$ is

$$(5.1) \quad gg' = (MM', XM' + X', k + k' + XM'JX'^t).$$

We shall also consider the restricted real Jacobi group $G_n^J(\mathbb{R})_0$, consisting only of elements of the form above, but $g = (M, X)$.

We consider also the Siegel-Jacobi upper half plane $\mathcal{X}_n^J := \mathcal{X}_n \times \mathbb{R}^{2n}$, where $\mathcal{X}_n = \text{Sp}(n, \mathbb{R})/\text{U}(n)$ is Siegel upper half plane realized as

$$\mathcal{X}_n := \{v \in M(n, \mathbb{C}) | v = s + ir, s, r \in M(n, \mathbb{R}), r > 0, s^t = s; r^t = r\}.$$

Let us consider an element $h = (g, l)$ in $G_n^J(\mathbb{R})_0$, i.e.

$$(5.2) \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(n, \mathbb{R}), \quad l = (n, m) \in \mathbb{R}^{2n},$$

and $v \in \mathcal{X}_n, u \in \mathbb{C}^n \equiv \mathbb{R}^{2n}$.

Now we consider the partial Cayley transform [10] $\Phi : \mathcal{X}_n^J \rightarrow \mathcal{D}_n^J, \Phi(v, u) = (W, z)$

$$(5.3a) \quad W = (v - i\mathbb{I}_n)(v + i\mathbb{I}_n)^{-1},$$

$$(5.3b) \quad z = 2i(v + i\mathbb{I}_n)^{-1}u,$$

with the inverse partial Cayley transform $\Phi^{-1} : \mathcal{D}_n^J \rightarrow \mathcal{X}_n^J, \Phi^{-1}(W, z) = (v, u)$

$$(5.4a) \quad v = i(\mathbb{I}_n - W)^{-1}(\mathbb{I}_n + W),$$

$$(5.4b) \quad u = (\mathbb{I}_n - W)^{-1}z.$$

Let us now define $\Theta : G_n^J(\mathbb{R})_0 \rightarrow G_n^J, \Theta(h) = h_*, h = (g, n, m), h_* = (g_{\mathbb{C}}, \alpha)$. We shall verify that (see also [66, 12])

Proposition 2. Θ is an group isomorphism and the action of G_n^J on \mathcal{D}_n^J is compatible with the action of $G_n^J(\mathbb{R})_0$ on \mathcal{X}_n^J through the biholomorphic partial Cayley transform (5.3), i.e. if $\Theta(h) = h_*$, then $\Phi h = h_* \Phi$. More exactly, if the action of G_n^J on \mathcal{D}_n^J is given by (3.9b), (3.9c), then the action of $G_n^J(\mathbb{R})_0$ on \mathcal{X}_n^J is given by $(g, l) \times (v, u) \rightarrow (v_1, u_1) \in \mathcal{X}_n^J$, where

$$(5.5a) \quad v_1 = (av + b)(cv + d)^{-1} = (vc^t + d^t)^{-1}(va^t + b^t);$$

$$(5.5b) \quad u_1 = (vc^t + d^t)^{-1}(u + vn + m).$$

The matrices g in (5.2) and $g_{\mathbb{C}}$ in (2.19) are related by (2.11), (2.13), while $\alpha = m + in$, $m, n \in \mathbb{R}^n$.

Proof. We introduce for W in (3.9b) its expression from (5.3a), and we get

$$W_1 = [(p + q)v + i(q - p)][(\bar{q} + \bar{p})v + i(\bar{p} - \bar{q})]^{-1}.$$

The expression of W_1 is introduced in the inverse (5.4a) of (5.3a) for v_1 . Use is made of (2.13) and the first equality in (5.5a) is obtained. The second one is a consequence of the symmetry of v , and it can be directly checked up with equations (2.10).

For the second assertion, we start with (5.3b), $2iu_1 = (v_1 + i\mathbb{I}_n)z_1$, in which we introduce the expression (3.9c) for z_1 . But with (5.4a), $v_1 + i\mathbb{I}_n = 2i(\mathbb{I}_n - W_1)^{-1}$, so we get

$$u_1 = (\mathbb{I}_n - W_1)^{-1}(Wq^* + p^*)[2i(v + i\mathbb{I}_n)^{-1}u + \alpha - W\bar{\alpha}].$$

In the above expression we write W_1 as function of W with the linear fractional transform (3.9b) and express W as function of v with the Cayley transform (5.3a). We replace $\alpha = m + in$, $m, n \in \mathbb{R}^n$, and express the matrix elements of $g_{\mathbb{C}}$ in function of the matrix elements of g via the relations (2.13) and we get also formula (5.5b). \square

Proposition 3. *The partial Cayley transform is a Kähler homogeneous diffeomorphism, $\Phi^*\omega_n = \omega'_n = \omega_n \circ \Phi$, i.e. under the transform (5.3), the Kähler two-form (3.12) on \mathcal{D}_n^J , G_n^J -invariant under the action (3.9b), (3.9c), becomes the Kähler two-form ω'_n (5.6) on \mathcal{X}_n^J , $G_n^J(\mathbb{R})_0$ -invariant to the action (5.5)*

$$(5.6) \quad -i\omega'_n = \frac{k}{2}\text{Tr}(H \wedge \bar{H}) + \frac{2}{i}\text{Tr}(G^t D \wedge \bar{G}), \quad \text{where} \\ D = (\bar{v} - v)^{-1}, H = D dv; G = du - dvD(\bar{u} - u).$$

Proof. The expression (5.6) was deduced firstly in [9]. Here we shortly indicate how to check up the group invariance. We calculate only the second term in the sum (5.6) because the first one is well known.

Differentiating (5.5b), we get

$$du_1 = (vc^t + d^t)^{-1}[du + dv(cv + d)^{-1}(dn - c(u + m))].$$

Now we calculate $\Psi = (v_1 - \bar{v}_1)^{-1}(u_1 - \bar{u}_1)$. Starting from (5.5a) and taking into account (2.10), we have

$$v_1 - \bar{v}_1 = (\bar{v}c^t + d^t)^{-1}(v - \bar{v})(cv + d)^{-1}.$$

Using

$$(\bar{v}c^t + d^t)(vc^t + d^t)^{-1} - \mathbb{I}_n = (\bar{v} - v)c^t(vc^t + d^t)^{-1}, \\ (\bar{v}c^t + d^t)(vc^t + d^t)^{-1}v - \bar{v} = (v - \bar{v})[\mathbb{I}_n + c^t(d^t)^{-1}v]^{-1},$$

we get

$$\Psi = dn - cm + (cv + d)(v - \bar{v})^{-1}[(\bar{v}c^t + d^t)(vc^t + d^t)^{-1}u - \bar{u}].$$

Taking into account (2.10a) in the differential of (5.5b), we get

$$dv_1 = (vc^t + d^t)dv(cv + d)^{-1},$$

and we find

$$G_1 = (vc^t + d^t)^{-1}(du + dv\Xi),$$

where

$$\begin{aligned}\Xi &= -(v - \bar{v})^{-1}Y + (v - \bar{v})^{-1}\bar{u}, \\ Y &= (v - \bar{v})(cv + d)^{-1}c + (\bar{v}c^t + d^t)(vc^t + d^t)^{-1}.\end{aligned}$$

Using relations of the type $v(cv + d)^{-1}c = (\mathbb{I}_n + vd^{-1}c)^{-1}vd^{-1}c$, it can be shown that $Y = \mathbb{I}_n$, and we get $G_1 = (vc^t + d^t)^{-1}G$, and the invariance of the second term in formula (5.6) is proved. Then the invariance of ω'_n under the action of (5.5) follows. \square

ω'_n given by (5.6) is the “ n ”-dimensional generalization of Berndt-Kähler two-form ω'_1 . In §37 in [36] Kähler calls \mathcal{X}_1^J *Phasenraum der Materie*, v is *Pneuma*, u is *Soma*.

5.1. Comparison with Yang’s results. J.-H. Yang [65]-[68] considers the Siegel-Jacobi space of degree n and order m , $\mathbf{H}_{n,m} = \mathcal{X}_n \times \mathbb{C}^{mn}$, the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{(\lambda, \mu, \kappa) | \lambda, \mu \in M_{mn}(\mathbb{R}), \kappa \in M_m(\mathbb{R})\},$$

and the Jacobi group $G^J = \mathrm{Sp}(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$, with the multiplication law

$$(M_0, (\lambda_0, \mu_0, \kappa_0)) \cdot (M, (\lambda, \mu, \kappa)) = (M_0M, (\tilde{\lambda}_0 + \lambda, \tilde{\mu}_0 + \mu, \kappa_0 + \kappa + \tilde{\lambda}_0\mu^t - \tilde{\mu}_0\lambda^t)),$$

where $(\tilde{\lambda}_0, \tilde{\mu}_0) = (\lambda_0, \mu_0)M$. G^J acts transitively on the Siegel-Jacobi space $\mathbf{H}_{n,m}$ by

$$(M, (\lambda, \mu, \kappa))(\Omega, Z) = (M \circ \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}),$$

and $G^J/K^J \cong \mathbf{H}_{n,m}$ is a nonreductive complex manifold, where $K^J = \mathrm{U}(n) \times \mathrm{Sym}(n, \mathbb{R})$. Now we identify (v, u) in our partial Cayley transform (5.5) with Yang’s partial Cayley transform (11) in [67],

$$(5.7) \quad \Omega = i(\mathbb{I}_n + W)(\mathbb{I}_n - W)^{-1}; Z = 2i\eta(\mathbb{I}_n - W)^{-1};$$

$$(5.8) \quad (v, u) \leftrightarrow (\Omega, \frac{Z^t}{2i}) \quad \text{and} \quad (W, z) \leftrightarrow (W, \eta^t).$$

Remark 2. The Kähler two-form in the case $m = 1$ in Theorem 1 in [67] is the Kähler two-form on \mathcal{X}_n^J (5.6), while the Kähler two-form on \mathcal{D}_n^J (3.12) is the corresponding one given in Theorem 6 in [67].

Proof. We use also Yang’s notation $\Omega = X + iY$; $Z = U + iV$ and we express our (5.6) in Yang’s notation as

$$\begin{aligned}D &= -\frac{1}{2i}Y^{-1}; \quad H = (-2iY)^{-1}d\Omega; \quad G = \frac{dZ^t}{2i} - d\Omega Y^{-1}V^t, \\ -iw'_n &= \frac{k}{8}\mathrm{Tr}(Y^{-1}d\Omega \wedge Y^{-1}d\bar{\Omega}) + \frac{1}{8}\mathrm{Tr}[(dZ - VY^{-1}d\Omega)Y^{-1} \wedge (d\bar{Z}^t - d\bar{\Omega}Y^{-1}V^t)], \\ -iw'_n &= \frac{k}{8}\mathrm{Tr}(Y^{-1}d\Omega \wedge Y^{-1}d\bar{\Omega}) + \frac{1}{8}\mathrm{Tr}(dZY^{-1} \wedge d\bar{Z}^t) + \frac{1}{8}\mathrm{Tr}(VY^{-1}d\Omega Y^{-1} \wedge d\bar{\Omega}Y^{-1}V^t) \\ &\quad - \frac{1}{8}\mathrm{Tr}(dZY^{-1} \wedge d\bar{\Omega}Y^{-1}V^t) - \frac{1}{8}\mathrm{Tr}(VY^{-1}d\Omega Y^{-1} \wedge d\bar{Z}^t).\end{aligned}$$

The second term in the sum above reads

$$dZ_{ij}Y_{jk}^{-1} \wedge d\bar{Z}_{ki}^t = Y_{kj}^{-1}dZ_{ji} \wedge d\bar{Z}_{ik} = \mathrm{Tr}(Y^{-1}dZ^t \wedge d\bar{Z}),$$

i.e. the corresponding term in Yang’s formula and similarly for the other 3 terms in the sum. \square

6. THE FUNDAMENTAL CONJECTURE FOR THE SIEGEL-JACOBI DOMAINS

Let us remind the *fundamental conjecture for homogeneous Kähler manifolds* (Gindikin -Vinberg): *every homogenous Kähler manifold is a holomorphic fiber bundle over a homogenous bounded domain in which the fiber is the product of a flat homogenous Kähler manifold and a compact simply connected homogenous Kähler manifold.* The compact case was considered by Wang [61]; Borel [21] and Matsushima [45] have considered the case of a transitive reductive group of automorphisms, while Gindikin and Vinberg [60] considered a transitive automorphism group. We mention also the essential contribution of Piatetski-Shapiro in this field [22]. The complex version, in the formulation of Dorfmeister and Nakajima [24], essentially asserts that: *every homogenous Kähler manifold, as a complex manifold, is the product of a compact simply connected homogenous manifold (generalized flag manifold), a homogenous bounded domain, and \mathbb{C}^n/Γ , where Γ denotes a discrete subgroup of translations of \mathbb{C}^n .*

Proposition 4. *Under the homogeneous Kähler transform FC (6.1),*

$$(6.1) \quad \mathbb{C}^n \times \mathcal{D}_n \ni (\eta, W) \xrightarrow{FC} (z, W) \in \mathcal{D}_n^J, FC(\eta, W) = (z, W), \quad z = \eta - W\bar{\eta},$$

$$(6.2) \quad FC^{-1} : \eta = (\mathbb{I}_n - W\bar{W})^{-1}(z + W\bar{z}).$$

the Kähler two-form (3.12) on \mathcal{D}_n^J , G_n^J -invariant to the action (3.9b), (3.9c), becomes the Kähler two-form on $\mathcal{D}_n \times \mathbb{C}^n$, $FC^\omega_n = \omega_{n,0}$,*

$$(6.3) \quad -i\omega_{n,0} = \frac{k}{2}\text{Tr}(B \wedge \bar{B}) + \text{Tr}(d\eta^t \wedge d\bar{\eta}),$$

invariant to the G_n^J -action on $\mathcal{D}_n \times \mathbb{C}^n$, $(g, \alpha) \cdot (\eta, W) \rightarrow (\eta_1, W_1)$, with W_1 given in (3.9c) and

$$(6.4) \quad \eta_1 = p(\eta + \alpha) + q(\bar{\eta} + \bar{\alpha}).$$

Under the homogenous Kähler transform

$$(6.5) \quad FC_1^{-1} : \eta = (\bar{v} - i\mathbb{I}_n)(\bar{v} - v)^{-1}(v - i\mathbb{I}_n)[(v - i\mathbb{I}_n)^{-1}u - (\bar{v} - i\mathbb{I}_n)^{-1}\bar{u}].$$

the Kähler two-form (5.6) becomes a Kähler two-form on $\mathcal{X}_n \times \mathbb{C}^n$, $FC_1^\omega'_n = \omega'_{n,0}$,*

$$(6.6) \quad -i\omega'_{n,0} = \frac{k}{2}\text{Tr}(H \wedge \bar{H}) + \text{Tr}(d\eta^t \wedge d\bar{\eta}), \quad H = (\bar{v} - v)^{-1}dv.$$

The inverse transform of (6.5) is

$$(6.7) \quad FC_1 : u = \frac{1}{2i}[(v + i\mathbb{I}_n)\eta - (v - i\mathbb{I}_n)\bar{\eta}].$$

The Kähler two-form (6.6) is invariant to the action $G_n^J(\mathbb{R})_0$ on $\mathcal{X}_n \times \mathbb{C}^n$, $(g, \alpha) \times (v, \eta) \rightarrow (v_1, \eta_1)$, where g has the form (5.2), v_1 is given by (5.5a), while

$$(6.8) \quad \eta_1 = \frac{1}{2}(\eta + \alpha)[a + d + i(b - c)] + \frac{1}{2}(\bar{\eta} + \bar{\alpha})[a - d - i(b + c)].$$

Proof. Following [36], [26], we introduce the variables $P, Q \in \mathbb{R}^n$ such that $u = vP + Q$, where $(u, v) \in \mathbb{C}^n \times \mathcal{X}_n$ are local coordinates on the Siegel-Jacobi upper-half plane \mathcal{X}_n^J . Using (5.4b), we have

$$u = vP + Q = (\mathbb{I}_n - W\bar{W})^{-1}z$$

and we introduce in formula above for v the expression given by (5.4a). We get $z = \eta - W\bar{\eta}$, where $\eta = P + iQ$ has appeared already in (3.9e) and (3.12). For A in (3.12) we get $A = d\eta - W d\bar{\eta}$. In (3.12), we make the transform (6.1). Also, from (6.1) and (5.3a), we have (3.9e), i.e. (6.2).

We use the relation $M - W\bar{M}\bar{W} = \mathbb{I}_n$ for the terms of the type $d\eta_i^t \wedge d\bar{\eta}_j$, the symmetry of the matrices $\bar{M}\bar{W}$ for the terms of the type $d\bar{\eta}_i^t \wedge d\bar{\eta}_j$ and $W\bar{M}$ for the terms of the type $d\eta_i^t \wedge d\eta_j$, and we get for $\omega_{n,0} = \omega_n \circ FC$ the expression given by (6.3).

Now we calculate the action of G_n^J on $\mathbb{C}^n \times \mathcal{D}_n$ induced from the action (3.9b), (3.9c) on \mathcal{D}_n^J applying the FC transform (6.1). We want to find η_1 from (6.2) with (W_1, z_1) given by (3.9b), (3.9c),

$$(6.9) \quad (\mathbb{I}_n - W_1\bar{W}_1)\eta_1 = z_1 + W_1\bar{z}_1.$$

Firstly, with (2.20), we calculate the lhs of (6.9) as

$$(6.10) \quad \mathbb{I}_n - W_1\bar{W}_1 = (Wq^* + p^*)^{-1}(\mathbb{I}_n - W\bar{W})(q\bar{W} + p)^{-1},$$

where p, q are components of the matrix $g \in \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ as in (2.19).

Now we introduce the value

$$z_1 = (Wq^* + p^*)^{-1}[\eta + \alpha - W(\bar{\eta} + \bar{\alpha})],$$

in rhs of (6.9), and we write it as

$$(6.11) \quad \begin{aligned} z_1 + W_1\bar{z}_1 &= (Wq^* + p^*)^{-1}Y, \quad \text{where} \\ Y &= \mathcal{P}(\eta + \alpha) + \mathcal{Q}(\bar{\eta} + \bar{\alpha}), \\ \mathcal{P} &= \mathbb{I}_n - (q^t + Wp^t)(\bar{W}b^t + a^t)^{-1}\bar{W}, \\ \mathcal{Q} &= -W + (q^t + Wp^t)(\bar{W}b^t + a^t)^{-1}. \end{aligned}$$

Using the successively the second relation in (2.20a), after an easy but long calculation, we find

$$(6.12) \quad \begin{aligned} \mathcal{P} &= (\mathbb{I}_n - W\bar{W})(p + q\bar{W})p, \\ \mathcal{Q} &= (\mathbb{I}_n - W\bar{W})(p + q\bar{W})q. \end{aligned}$$

Combining (6.11)-(6.12), we get for η_1 the value given in (6.4).

In order to verify the invariance of the Kähler two-form (6.3) to the action (6.4), we have to check up that

$$(6.13) \quad \mathrm{Tr}(d\eta_1^t \wedge d\bar{\eta}_1) = \mathrm{Tr}(d\eta^t \wedge d\bar{\eta}),$$

which is true because of the first relation in (2.20b).

With (5.4a), we get

$$(6.14) \quad (\mathbb{I}_n - W\bar{W})^{-1} = \frac{1}{2i}(\bar{v} - i\mathbb{I}_n)(\bar{v} - v)^{-1}(v + i\mathbb{I}_n).$$

We introduce (6.14), (5.3b) and (5.3a) in (3.9e) and we get (6.5).

(6.7) is obtained introducing in (5.3b) the expression (6.1) with W given by (5.3a).

Finally, (6.8) is a consequence of (6.4) and (2.14). Alternatively, the invariance (6.13), where η_1 has the expression (6.8), can be checked up directly, taking into account that

the matrices a, b, c, d appearing in (6.8) are the components of $g \in \text{Sp}(n, \mathbb{R})$ as in (5.2), and consequently verify the conditions (2.10). \square

We recall that in the case $n = 1$, (6.6) appears in the paper [36] of Erich Kähler as equation (3) in §38.

Corollary 1. *Under the FC-change of coordinates (6.1), $x = \eta - V\bar{\eta}$, $y = \xi - W\bar{\xi}$, the reproducing kernel (3.13) becomes $\mathcal{K} = K \circ FC$,*

$$(6.15) \quad \begin{aligned} \mathcal{K}(\bar{\eta}, \bar{V}; \xi, W) &= (\det U)^{k/2} \exp \mathcal{F}, \quad \text{where} \\ 2\mathcal{F} &= \mathcal{F}_0 + \Delta\mathcal{F}; \\ \mathcal{F}_0 &= \bar{\xi}^t \xi + \bar{\eta}^t \eta - \bar{\xi}^t W \bar{\xi} - \eta^t \bar{V} \eta, \\ \Delta\mathcal{F} &= (\bar{\xi}^t - \xi^t \bar{V})U(\xi - W\bar{\xi}) + (\bar{\eta}^t - \eta^t \bar{V})U(-\xi + W\bar{\xi}); \xi = \eta - \xi. \end{aligned}$$

In particular, for $\xi = \eta, V = W$, we have $\Delta\mathcal{F} = 0$, and

$$(6.16) \quad \mathcal{K} = \det(M)^{\frac{k}{2}} \exp(\mathcal{F}), \quad \text{where } \mathcal{F} = \bar{\eta}^t \eta - \frac{1}{2} \eta^t \bar{W} \eta - \frac{1}{2} \bar{\eta}^t W \bar{\eta},$$

and the scalar product (3.19) becomes

$$(6.17) \quad \begin{aligned} (\phi, \psi) &= \Lambda_n \int_{\eta \in \mathbb{C}^n; \mathbb{I}_n - W\bar{W} > 0} \bar{f}_\phi(\eta, W) f_\psi(\eta, W) \rho_2 \, d\eta \, dW, \\ \rho_2 &= \det(\mathbb{I}_n - W\bar{W})^q \exp(-\mathcal{F}), \, q = k/2 - n - 1, \end{aligned}$$

with \mathcal{F} given by (6.16).

Also we have the relation

$$-\frac{\partial F}{\partial w_{ij}} = \frac{\partial \mathcal{F}}{\partial w_{ij}} = -\bar{\eta}_i \bar{\eta}_j + \frac{1}{2} \bar{\eta}_i \bar{\eta}_j \delta_{ij}, \quad \text{or } \nabla F = -\nabla \mathcal{F} = \frac{1}{2} \bar{\eta} \otimes \bar{\eta}.$$

Proof. Formula (6.15) is obtained by brute force calculation, using the relation $UW\bar{V} = U - \mathbb{I}_n$.

In the expression (3.14), we make the change of variables (6.1), and we get easily the expression of \mathcal{F} given in (6.16).

In order to get the factor ρ_2 in the expression (6.17), we firstly note that

$$(z^t, \bar{z}^t) = (\eta^t, \bar{\eta}^t) \begin{pmatrix} \mathbb{I}_n & -\bar{W} \\ -W & \mathbb{I}_n \end{pmatrix}.$$

Now, we observe that if we have a linear transformation $y = Cx$ of the column n -vectors y and x , then

$$y_1 \wedge \cdots \wedge y_n = \det(C) x_1 \wedge \cdots \wedge x_n.$$

We apply the formula

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - CA^{-1}B).$$

\square

7. CLASSICAL MOTION AND QUANTUM EVOLUTION ON SIEGEL-JACOBI DOMAINS

Let $M = G/H$ be a homogeneous manifold with the G -invariant Kähler two-form ω (3.3). The energy function \mathcal{H} (the classical Hamiltonian, or the covariant symbol) attached to the quantum Hamiltonian \mathbf{H} is [16]

$$\mathcal{H}(z, \bar{z}) = \langle e_{\bar{z}}, e_{\bar{z}} \rangle^{-1} \langle e_{\bar{z}} | \mathbf{H} | e_{\bar{z}} \rangle.$$

Passing on from the dynamical system problem in the Hilbert space \mathfrak{H} to the corresponding one on M is called sometimes *dequantization*, and the dynamical system on M is a classical one [4, 5]. Following Berezin [15, 17], the motion on the classical phase space can be described by the local equations of motion

$$(7.1) \quad \dot{z}_\alpha = i \{ \mathcal{H}, z_\alpha \}, \quad \alpha \in \Delta_+,$$

where the Poisson bracket is defined as

$$\{f, g\} = \sum_{\alpha, \beta \in \Delta_+} (\mathcal{G}^{-1})_{\alpha, \beta} \left(\frac{\partial f}{\partial z_\alpha} \cdot \frac{\partial g}{\partial \bar{z}_\beta} - \frac{\partial f}{\partial \bar{z}_\alpha} \cdot \frac{\partial g}{\partial z_\beta} \right),$$

$f, g \in C^\infty(M)$ and $(\mathcal{G}^{-1})_{\alpha, \beta}$ are the matrix elements of the inverse of the matrix \mathcal{G} defined in (3.4).

The classical equations of motion (7.1) on the manifold M can be written down as

$$(7.2) \quad i \begin{pmatrix} \mathbb{O}_n & \mathcal{G} \\ -\bar{\mathcal{G}} & \mathbb{O}_n \end{pmatrix} \begin{pmatrix} \dot{z} \\ \dot{\bar{z}} \end{pmatrix} = - \begin{pmatrix} \partial/\partial z \\ \partial/\partial \bar{z} \end{pmatrix} \mathcal{H}.$$

We consider an algebraic Hamiltonian linear in the generators X_λ of the group of symmetry G

$$(7.3) \quad H = \sum_{\lambda \in \Delta} \epsilon_\lambda X_\lambda,$$

and we also consider the associated operator \mathbf{H} . The energy function associated to the Hamiltonian (7.3) is

$$\mathcal{H}(z, \bar{z}) = \sum_{\lambda \in \Delta} \epsilon_\lambda \frac{(e_{\bar{z}}, \mathbf{X}_\lambda e_{\bar{z}})}{(e_{\bar{z}}, e_{\bar{z}})}.$$

Suppose that $\mathbf{X}_\lambda \in \mathfrak{D}_1$, i.e. the differential action corresponding to the operator \mathbf{X}_λ in (7.3) can be expressed in a local system of coordinates as a holomorphic first order differential operator with polynomial coefficients of the type (4.3)

$$(7.4) \quad \mathbb{X}_\lambda = P_\lambda + \sum_{\beta \in \Delta_+} Q_{\lambda, \beta} \partial_\beta, \quad \lambda \in \Delta.$$

If $P_\lambda, Q_{\lambda, \beta}$ are the polynomials attached to the operator \mathbf{X}_λ in (7.4), let $\tilde{P}_\lambda, \tilde{Q}_{\lambda, \beta}$ be the polynomials attached to the operator $\mathbf{X}_\lambda^\dagger$ for the generator $X_\lambda \in \mathfrak{g}$ appearing in (7.3).

With (3.4), we get

$$\frac{\partial \mathcal{H}}{\partial \bar{z}_\beta} = \sum_{\lambda, \gamma} \epsilon_\lambda Q_{\lambda, \gamma} \mathcal{G}_{\gamma, \beta},$$

and the classical motion generated by the linear Hamiltonian (7.3) is given by the equations of motion on $M = G/H$ [4, 5]:

$$(7.5) \quad i\dot{z}_\alpha = \sum_{\lambda \in \Delta} \epsilon_\lambda Q_{\lambda, \alpha}, \quad \alpha \in \Delta_+.$$

Similarly, the equations of motion on $M = G/H$ determined by a Hamiltonian

$$(7.6) \quad \mathbf{H} = \sum_{\lambda \in \Delta} \epsilon_\lambda \mathbf{X}_\lambda^\dagger,$$

are

$$(7.7) \quad i\dot{\tilde{z}}_\alpha = \sum_{\lambda \in \Delta} \epsilon_\lambda \tilde{Q}_{\lambda, \alpha}, \quad \alpha \in \Delta_+.$$

We look also for the solutions of the Schrödinger equation attached to the Hamiltonian \mathbf{H} (7.3)

$$(7.8) \quad \mathbf{H}\psi = i\dot{\psi}, \quad \text{where } \psi = e^{i\varphi} < e_z, e_z >^{-1/2} e_{\bar{z}}.$$

Summarizing, in the conventions at the beginning of §4 and with the observation (4.7), we formulate the following (see also [5], [52])

Proposition 5. *On the homogenous manifold $M = G/H$ on which the generators $X_\lambda \in \mathfrak{g}$ admit a holomorphic representation as in (7.4), the classical motion and the quantum evolution generated by the linear Hamiltonian (7.3) are given by the same equation of motion (7.5). The phase φ in (7.8) is given by the sum $\varphi = \varphi_D + \varphi_B$ of the dynamical and Berry phase,*

$$(7.9a) \quad \varphi_D = - \int_0^t \mathcal{H}(t) dt, \quad \text{where}$$

$$(7.9b) \quad \mathcal{H}(t) = \sum_{\lambda \in \Delta} \epsilon_\lambda \frac{(e_{\bar{z}}, \mathbf{X}_\lambda e_{\bar{z}})}{(e_{\bar{z}}, e_{\bar{z}})}$$

$$(7.9c) \quad = \sum_{\lambda \in \Delta} \epsilon_\lambda (\tilde{P}_\lambda + \sum_{\beta \in \Delta_+} \tilde{Q}_{\lambda, \beta} \partial_\beta \ln \langle e_z, e_z \rangle)$$

$$(7.9d) \quad = \sum_{\lambda \in \Delta} \epsilon_\lambda \tilde{P}_\lambda + i \sum_{\beta \in \Delta_+} \dot{\tilde{z}}_\beta \partial_\beta \ln \langle e_{\bar{z}}, e_{\bar{z}} \rangle ;$$

$$(7.9e) \quad \begin{aligned} \varphi_B &= -\Im \int_0^t \langle e_{\bar{z}}, e_{\bar{z}} \rangle^{-1} \langle e_{\bar{z}} | d | e_{\bar{z}} \rangle \\ &= \frac{i}{2} \int_0^t \sum_{\alpha \in \Delta_+} (\dot{z}_\alpha \partial_\alpha - \dot{\tilde{z}}_\alpha \bar{\partial}_\alpha) \ln \langle e_{\bar{z}}, e_{\bar{z}} \rangle . \end{aligned}$$

7.1. Equations of motion. Now we consider a Hamiltonian linear in the generators of the group G_n^J

$$(7.10) \quad \mathbf{H} = \epsilon_i \mathbf{a}_i + \bar{\epsilon}_i \mathbf{a}_i^\dagger + \epsilon_{ij}^0 \mathbf{K}_{ij}^0 + \epsilon_{ij}^- \mathbf{K}_{ij}^- + \epsilon_{ij}^+ \mathbf{K}_{ij}^+.$$

The hermiticity condition imposes to the matrices of coefficients $\epsilon_{0,\pm} = (\epsilon^{0,\pm})_{i,j=1,\dots,n}$ the restrictions

$$(7.11) \quad \epsilon_0^\dagger = \epsilon_0; \quad \epsilon_- = \epsilon_-^t; \quad \epsilon_+ = \epsilon_+^t; \quad \epsilon_+^\dagger = \epsilon_-.$$

It is useful to introduce the matrices $m, n, p, q \in M(n, \mathbb{R})$ such that

$$(7.12) \quad \epsilon_- = m + in, \quad \epsilon_0^t/2 = p + iq; \quad p^t = p; \quad m^t = m; \quad n^t = n; \quad q^t = -q.$$

We shall describe the dynamics on the Siegel-Jacobi ball (space) \mathcal{D}_n^J (respectively, \mathcal{X}_n^J) determined by the linear Hamiltonian (7.10), (7.11) and study the effect of the FC (FC_1) transform on the equations of motion. We introduce some notation. Let $W \in M(n, \mathbb{C})$ be coordinates on a homogenous manifolds $M = G/H$ - below M will be one of the Siegel-Jacobi domains \mathcal{D}_n or \mathcal{X}_n - and let $z \in \mathbb{C}^n$. We consider a matrix Riccati equation on the manifold M and a linear differential equation in z

$$(7.13a) \quad \dot{W} = AW + WD + B + WCW, \quad A, B, C, D \in M(n, \mathbb{C});$$

$$(7.13b) \quad \dot{z} = M + Nz; \quad M = E + WF; \quad N = A + WC, \quad E, F \in C^n.$$

Proposition 6. *The classical motion and quantum evolution generated by the linear hermitian Hamiltonian (7.10), (7.11) are described by first order differential equations: a) on \mathcal{D}_n^J , $(z, W) \in \mathbb{C}^n \times \mathcal{D}_n$ verifies (7.13), with coefficients*

$$(7.14a) \quad A_c = -\frac{i}{2}\epsilon_0^t, \quad B_c = -i\epsilon_-, \quad C_c = -i\epsilon_+, \quad D_c = A_c^\dagger;$$

$$(7.14b) \quad E_c = -i\epsilon, \quad F_c = -i\bar{\epsilon}.$$

b) on \mathcal{X}_n^J , $(u, v) \in \mathbb{C}^n \times \mathcal{X}_n$, verifies (7.13), with coefficients

$$(7.15a) \quad A_r = n + q, \quad B_r = m - p, \quad C_r = -(m + p), \quad D_r = n - q;$$

$$(7.15b) \quad E_r = \Im \epsilon; \quad F_r = -\Re \epsilon.$$

c) under the FC transform (6.1), the differential equations in the variables $\eta \in \mathbb{C}^n$, $W \in \mathcal{D}_n$ become independent: W verifies (7.13a) with coefficients (7.14a) and η verifies

$$(7.16) \quad i\dot{\eta} = \epsilon + \epsilon_- \bar{\eta} + \frac{1}{2}\epsilon_0^t \eta, \quad \eta \in \mathbb{C}^n.$$

d) under the FC_1 transform, the equations in the variables $\eta \in \mathbb{C}^n$, $v \in \mathcal{X}_n$ become independent: η verifies (7.16), while v verifies (7.13a) with coefficients (7.15a).

Proof. Firstly, we proof (7.14). With (4.8a) in Lemma 1, we get from (7.5) the equations of motion for $z \in \mathbb{C}^n$:

$$i\dot{z}_\alpha = \epsilon_i \delta_{i\alpha} + \bar{\epsilon}_i w_{i\alpha} + \frac{\epsilon_{kl}^+}{2} (z_k w_{il} + z_l w_{ik}) \delta_{i\alpha} + \frac{1}{2} \epsilon_{kl}^0 z_k \delta_{l\alpha}.$$

The equations of motion (7.5) for w_{pq} , $W = (w_{pq})_{p,q=1,\dots,n}$, can be written down as

$$i\dot{w}_{pq} = \epsilon_{kl} Q_{kl,pq}.$$

With (4.8b), (4.8c) in Lemma 1 and (4.5), we have for $Q_{kl,pq}$ -s the expressions:

$$\begin{aligned} \mathbf{K}_{kl}^0 &\rightarrow Q_{kl,pq}^0 = w_{kp}\chi_{pl}\delta_{lq} + w_{kq}\chi_{ql}\delta_{lp} - w_{kl}\delta_{qp}\delta_{lp}, \\ \mathbf{K}_{kl}^+ &\rightarrow Q_{kl,pq}^+ = w_{qk}w_{pl}\chi_{pq} + w_{pk}w_{ql}\chi_{qp} - w_{pk}w_{pl}\delta_{pq}, \\ \mathbf{K}_{kl}^- &\rightarrow Q_{kl,pq}^- = \chi_{kl}(\delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp} - \delta_{kl}\delta_{pq}\delta_{kp}). \end{aligned}$$

Explicitly, the differential equations for $(W, z) \in \mathcal{D}_n^J$ are

$$(7.17a) \quad i\dot{W} = \epsilon_- + (W\epsilon_0)^s + W\epsilon_+W, \quad W \in \mathcal{D}_n,$$

$$(7.17b) \quad i\dot{z} = \epsilon + W\bar{\epsilon} + \frac{1}{2}\epsilon_0^t z + W\epsilon_+z, \quad z \in \mathbb{C}^n,$$

and we have proved a).

Now we prove b). Firstly, we prove (7.15a).

With the Cayley transform (5.3a), we find for \dot{W} the value

$$2i(v + i\mathbb{I}_n)^{-1}\dot{v}(v + i\mathbb{I}_n)^{-1} = A_cW + WD_c + B_c + WC_cW.$$

In the expression above we replace the value of W as function of v as given by (5.3a), and we find a matrix Riccati equation on \mathcal{X}_n in v of the form (7.13a), where the matrix coefficients $A_r, D_r \in M(n, \mathbb{R})$ are expressed in function of the coefficients A_c, D_c as

$$(7.18) \quad \begin{aligned} A_r &= \frac{1}{2}(A_c - D_c + B_c - C_c), \quad B_r = \frac{1}{2i}(A_c + D_c - B_c - C_c), \\ C_r &= \frac{1}{2i}(A_c + D_c + B_c + C_c), \quad D_r = \frac{1}{2}(-A_c + D_c + B_c - C_c). \end{aligned}$$

With (7.12), we find the values given by (7.15a).

Now we proof (7.15b). We differentiate (5.3b) and, with (7.14b), we get successively

$$\begin{aligned} -2\dot{u} &= \dot{v}(iz) + (v + i\mathbb{I}_n)(i\dot{z}) \\ &= -2\dot{v}(v + i\mathbb{I}_n)^{-1}u + (v + i\mathbb{I}_n) \times \\ &\quad \times \{ \epsilon + (v + i\mathbb{I}_n)^{-1}(v - i\mathbb{I}_n)\bar{\epsilon} + [\frac{\epsilon_0^t}{2} + (v + i\mathbb{I}_n)^{-1}(v - i\mathbb{I}_n)\epsilon_+]2i(v + i\mathbb{I}_n)^{-1} \}u \\ &= (v + i\mathbb{I}_n)\epsilon + (v - i\mathbb{I}_n)\bar{\epsilon} + T(v + i\mathbb{I}_n)^{-1}u, \end{aligned}$$

where

$$T = -2\dot{v} + i(v + i\mathbb{I}_n)\epsilon_0^t + 2i(v - i\mathbb{I}_n)\epsilon_+.$$

With (7.15a) we get for T the value

$$T = [v(\epsilon_0^s + \epsilon_+ + \epsilon_-) + i(-\epsilon_0^a + \epsilon_- - \epsilon_+)](v + i\mathbb{I}_n),$$

and we obtain (7.15b). Explicitly, b) can be write down as

$$(7.19a) \quad \begin{aligned} -2\dot{v} &= \epsilon_0^s - (\epsilon_- + \epsilon_+) + iv(\epsilon_0^a + \epsilon_- - \epsilon_+) + \\ &\quad i(-\epsilon_0^a + \epsilon_- - \epsilon_+)v + v(\epsilon_0^s + \epsilon_- + \epsilon_+)v; \end{aligned}$$

$$(7.19b) \quad -2\dot{u} = v(\epsilon + \bar{\epsilon}) + i(\epsilon - \bar{\epsilon}) + [v(\epsilon_0^s + \epsilon_+ + \epsilon_-) + i(-\epsilon_0^s + \epsilon_- - \epsilon_+)]u.$$

Below we prove (7.16). We take the derivative of η in the FC^{-1} transform (6.2), and we get

$$\dot{\eta} = (\mathbb{I}_n - W\bar{W})^{-1}(\dot{W}\bar{W} + W\dot{\bar{W}})\eta + (\mathbb{I}_n - W\bar{W})^{-1}(\dot{z} + \dot{W}\bar{z} + W\dot{\bar{z}}).$$

Then we introduce for \dot{W} and \dot{z} the values from (7.17a) and respectively (7.17b) and we pass from z to η with (6.1):

$$\begin{aligned} i(\mathbb{I}_n - W\bar{W})\dot{\eta} &= \{[\epsilon_- + (W\epsilon_0)^s + W\epsilon_+W]\bar{W} - W[\bar{\epsilon}_- + (\bar{W}\bar{\epsilon}_0)^s + \bar{W}\bar{\epsilon}_+\bar{W}]\}\eta + \\ &\quad [\epsilon_- + (W\epsilon_0)^s + W\epsilon_+W](\bar{\eta} - \bar{W}\eta) - W[\bar{\epsilon}_- + \bar{W}\epsilon + (\frac{\bar{\epsilon}_0^t}{2} + \bar{W}\bar{\epsilon}_+)(\bar{\eta} - \bar{W}\eta)] + \\ &\quad \epsilon + W\bar{\epsilon} + (\frac{\epsilon_0^t}{2} + W\epsilon_+)(\eta - W\bar{\eta}). \end{aligned}$$

We calculate the coefficient of η as $\frac{1}{2}(\mathbb{I}_n - W\bar{W})\epsilon_0^t$, those of $\bar{\eta}$ is $(\mathbb{I}_n - W\bar{W})\epsilon_-$, and we get (7.16).

In order to avoid the longer calculation of introducing (7.15b) in (6.5), the motion in $(\eta, v) \in (\mathbb{C}^n, \mathcal{X}_n)$ at d) is obtained putting together (7.16) and (7.15a). \square

Note that starting with the *Hamiltonian* (7.10), *linear in the generators of the Jacobi group* G_n^J , the equation of motion (7.14a) ((7.15a)) on the Siegel ball \mathcal{D}_n (Siegel upper half plane \mathcal{X}_n) depends **only** on the generators of the group $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ (respectively, $\mathrm{Sp}(n, \mathbb{R})$).

7.2. Solution of the equations of motion. We solve the system of differential equations on the Siegel-Jacobi domains appearing in Proposition 6.

a. Firstly, we recall how **to solve the matrix Riccati equation** (7.13a) **by linearization**. If we proceed to the homogenous coordinates $W = XY^{-1}$, $X, Y \in M(n, \mathbb{C})$, a linear system of ordinary differential equations is attached to the matrix Riccati equation (7.13a) (cf. [42], see also [5])

$$(7.20) \quad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h \begin{pmatrix} X \\ Y \end{pmatrix}, \quad h = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix}.$$

Every solution of (7.20) is a solution of (7.13a), whenever $\det(Y) \neq 0$.

For the motion on \mathcal{D}_n , the matrix elements of h in (7.20), denoted h_c , are given in (7.14a), and the linear system of differential equations attached to the matrix Riccati equation (7.17a) is

$$(7.21) \quad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h_c \begin{pmatrix} X \\ Y \end{pmatrix}, \quad h_c = \begin{pmatrix} -i(\frac{\epsilon_0}{2})^t & -i\epsilon_- \\ i\epsilon_+ & i\frac{\epsilon_0}{2} \end{pmatrix}, \quad W = X/Y \in \mathcal{D}_n.$$

Due to the conditions (7.11), the matrix h_c has the form (2.15), and we conclude that $h_c \in \mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$. The action of the element $g \in \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ (2.19) on $W \in \mathcal{D}_n$ is given by the linear fractional transformation (3.9b).

Now we look at the matrix Riccati equation (7.13a) on the Siegel upper half plane \mathcal{X}_n in the symmetric variables v with coefficients $A_r - D_r$ given by (7.18) – or equation (7.19a). We associate to the matrix Riccati equation (7.19a) a linear system of first order differential equations of the type (7.20) with a matrix coefficients h_r

$$(7.22) \quad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h_r \begin{pmatrix} X \\ Y \end{pmatrix}, \quad h_r = \begin{pmatrix} A_r & B_r \\ -C_r & -D_r \end{pmatrix} = \begin{pmatrix} n+q & m-p \\ m+p & -n+q \end{pmatrix}, \quad v = X/Y \in \mathcal{X}_n.$$

Due to the hermiticity conditions (7.11), the matrix h_r verifies the conditions (2.9), because of (7.18), and we see that $h_r \in \mathfrak{sp}(n, \mathbb{R})$. The matrices h_r and h_c are related by relations of the type (2.16), (2.17), i.e. $h_c = (h_r)_{\mathbb{C}}$ and have the same eigenvalues.

Using general considerations of solving systems of first order linear differential equations [31] applied to $\mathfrak{sp}(n, \mathbb{R})$ [46] and the considerations above, we formulate

Remark 3. The linear system of first order differential equations (7.21) ((7.22)) associated to the matrix Riccati equations (7.17a) ((7.19a)) on \mathcal{D}_n (\mathcal{X}_n) describes the time-dependent vector field induced by the infinitesimal action of the group $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ (respectively, $\mathrm{Sp}(n, \mathbb{R})$). (7.21) ((7.22)) is a linear Hamiltonian system in the meaning of Meyer [46] or a canonical system in the sense of Yakulovich [63] (Hamiltonian system) (respectively, in the sense Yakulovich [63]) and $h_c = (h_r)_{\mathbb{C}}$.

The infinitesimal group action of $\mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$ (respectively, $\mathfrak{sp}(n, \mathbb{R})$) is given by the Lie algebras homomorphism

$$(7.23) \quad \nu_c : \mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}} \rightarrow \mathrm{Ham}(\mathcal{D}_n), \quad \nu_r : \mathfrak{sp}(n, \mathbb{R}) \rightarrow \mathrm{Ham}(\mathcal{X}_n).$$

The infinitesimal group action associated to (7.20) is given by

$$(7.24) \quad \nu \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} = -(B + AZ + ZD + ZCZ)_{im} \frac{\partial}{\partial w_{im}}.$$

Let

$$(7.25) \quad U(t, t_0) = \begin{pmatrix} U_1(t, t_0) & U_2(t, t_0) \\ U_3(t, t_0) & U_4(t, t_0) \end{pmatrix}$$

be the fundamental matrix of the ordinary differential equation (7.20), i.e. $\dot{U} = hU$ such that $U(t_0, t_0) = 1$, where $h = h_c$ ($h = h_r$) for the motion on \mathcal{D}_n (respectively, \mathcal{X}_n). Then the fundamental solution $U_c(t, t_0)$ ($U_r(t, t_0)$) corresponding to $h = h_c$ ($h = h_r$) is a $\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$ (respectively $\mathrm{Sp}(n, \mathbb{R})$)-matrix and

$$W(t, t_0) = [U_1(t, t_0)W(t_0) + U_2(t, t_0)][U_3(t, t_0)W(t_0) + U_4(t, t_0)]^{-1}$$

is the solution of equation (7.13a) with the initial condition $W(t_0, t_0) = W(t_0)$.

For selfcontainedness, we recall the terminology used in Remark 3. The systems of linear ordinary differential equations $\dot{z} = Az$, $A \in \mathfrak{sp}(n, \mathbb{R})$ appear in the context of linear Hamiltonian systems [46]. The eigenvalues of the Hamiltonian matrices are described in Remark 1 e). The Hamiltonian equations can be written as the system of ordinary differential equations

$$(7.26) \quad \dot{z} = J\nabla H,$$

where $z^t = (q^t, p^t)$, $q, p \in \mathbb{R}^n$, $H = H(t, q, p)$ is the Hamiltonian, and here $\nabla H = (\frac{\partial H}{\partial z_1}, \dots, \frac{\partial H}{\partial z_{2n}})$. The Hamiltonian system (7.26) is called a *Hamiltonian linear system* (cf [46], also called *canonical*, cf [63] p. 110), if the Hamiltonian H has the form

$$(7.27) \quad H = \frac{1}{2} z^t S z, \quad S \in M(2n, \mathbb{R}), \quad S = S^t,$$

and the system (7.26) is written as the system of linear ordinary differential equations

$$(7.28) \quad \dot{z} = JSz = Az, \quad A \in \mathfrak{sp}(n, \mathbb{R}), \quad S = S^t.$$

The equation $\dot{z} = Az$, $A \in \mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$, where A is expressed as in (2.18), is also called *Hamiltonian*, cf. [63] p. 111.

b. Now we discuss the linear system of the type (7.28) associated to the **matrix Riccati equation** (7.19a) **in the case of T -periodic coefficients**. Let $\Delta(t)$ be the fundamental solution of the equation (7.28) with periodic coefficients such $\Delta(0) = \mathbb{I}_{2n}$. Then $\Delta(t + T) = \Delta(t)\Delta(T)$, $\forall t \in \mathbb{R}$. We take over *Floquet-Lyapunov Theorem* and *Krein Gel'fand Theorem*, (cf. Proposition 4.2.1, Theorem 3.4.2 and Corollary 3.4.1 in [46]; see also Ch II in [63]), which we formulate as:

Remark 4. *The monodromy matrix $\Delta(T)$ is a nonsingular, symplectic matrix. The fundamental matrix solution of Hamiltonian equation (7.28) that satisfies $\Delta(0) = \mathbb{I}_{2n}$ is of the form $\Delta(t) = X(t) \exp(Kt)$, where $X(t)$ is symplectic and T -periodic and K is Hamiltonian. Real $X(t)$ and K can be found by taking $X(t)$ to be $2T$ -periodic if necessary. The change of variables $z = X(t)w$ transforms the periodic Hamiltonian system (7.28) to the constant Hamiltonian system*

$$(7.29) \quad \dot{w} = Kw, \quad K \in \mathfrak{sp}(n, \mathbb{R}).$$

The linear autonomous Hamiltonian system (7.29) is stable iff: (i) K it has only pure imaginary eigenvalues and (ii) K is diagonalizable (over the complex numbers).

The system (7.29) is parametrically stable if and only if: (i) All the eigenvalues of K are pure imaginary $\pm i\alpha_i$; (ii) K is nonsingular; (iii) The matrix K is diagonalizable over the complex numbers; (iv) The restriction of the Hamiltonian H to the linear space generated by $\pm i\alpha_j$ is positive or negative definite for each j .

c. Let us discuss **the solution of the matrix Riccati equation** (7.13a) **in the case of constant coefficients**. Let $\lambda_1, \dots, \lambda_{2n}$ be the eigenvalues of characteristic equation associated to the matrix h defined in (7.20)

$$(7.30) \quad \det(h - \lambda \mathbb{I}_{2n}) = 0.$$

Let

$$(7.31) \quad \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},$$

where Λ_1 (Λ_2) is a diagonal matrix with entries λ_i , $i = 1, \dots, n$ (respectively, $i = n + 1, \dots, 2n$).

Let us suppose that the matrix h has a *simple structure* (cf. Ch. III in [29] or h has *simple elementary divisors* cf. [63]) and let V be the *fundamental matrix* [29] of h (7.20), i.e.

$$(7.32) \quad Vh = \Lambda V.$$

Let us consider a partition of V into block form

$$(7.33) \quad V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix},$$

where $V_i, i = 1, \dots, 4 \in M(n, \mathbb{C})$.

Let $Q = Q(W, A, B, C, D)$ the matrix which appears in the rhs of (7.13a), and let us denote by Q_0 the value of Q at $W_0 = W(t = 0)$.

Remark 5. *The autonomous linear system (7.20) associated to the matrix Riccati equation (7.13a) has the standard solution*

$$(7.34) \quad \begin{pmatrix} X \\ Y \end{pmatrix} = e^{th} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}.$$

The solution W of the matrix Riccati equation (7.13a) has the Taylor expansion

$$W(t) = W_0 + tQ_0 + \frac{t^2}{2}[(A + W_0C)Q_0 + Q_0(D + CW_0)] + \dots$$

If the matrix h (7.20) has a simple structure, then the solution of the matrix Riccati equation (7.13a) with constant coefficients with the initial condition $W(0) = W_0$ is

$$(7.35a) \quad W = (V_1 - W'V_3)^{-1}(W'V_4 - V_2),$$

$$(7.35b) \quad W' = W'(t, W'_0) = e^{t\Lambda_1}W'_0e^{-t\Lambda_2},$$

$$(7.35c) \quad W'_0 = (V_1W_0 + V_2)(V_3W_0 + V_4)^{-1}.$$

Closed paths on the Siegel ball \mathcal{D}_n ($W(T) = W(0)$) can be obtained if the imaginary eigenvalues λ_i of the matrix h are rationally commensurable and different and T is the least common multiple of $2\pi|\lambda_m - \lambda_i|$, $i = 1, \dots, n$, $m = n+1, \dots, 2n$.

Suppose now we are in the case when the autonomous Hamiltonian system (7.21) is stable: the matrix h_r has $2n$ purely imaginary distinct eigenvalues and the matrix h_r can be put in the form

$$(7.36) \quad h_r = \begin{pmatrix} \mathbb{O}_n & d \\ -d & \mathbb{O}_n \end{pmatrix}, \quad d = \text{diag}(\alpha_1, \dots, \alpha_n).$$

Correspondingly, in (7.31) we get

$$(7.37) \quad \Lambda_1 = \text{id}, \quad \Lambda_2 = -\text{id}$$

which have to be introduced in (7.35) for the motion on \mathcal{X}_n in the case of the constant coefficients in (7.15a).

We also are in the case where the matrix h_c has $2n$ distinct pure imaginary eigenvalues and h_c can be put into the form

$$(7.38) \quad h_c = \begin{pmatrix} \text{id} & \mathbb{O}_n \\ \mathbb{O}_n & -\text{id} \end{pmatrix}, \quad d = \text{diag}(\alpha_1, \dots, \alpha_n),$$

with the same eigenvalues as in (7.37).

Proof. As was mentioned in Remark 3 a), the linear system of first order differential equations (7.20) is associated with the matrix Riccati equation (7.13a). A solution to (7.20) projects to a solution to (7.13a) via the map $\Psi(X, Y) = XY^{-1}$. The autonomous linear system (7.20) has the standard solution (7.34) [31]. We recall that if $A \in M(n, \mathbb{F})$, ($\mathbb{F} = \mathbb{R}$ or \mathbb{C}) has the minimal polynomial $\psi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \dots (\lambda - \lambda_s)^{m_s}$, where $\lambda_1, \dots, \lambda_s$ are characteristic roots of A , then

$$e^{At} = \sum_{k=1}^s [Z_{k1} + Z_{k2}t + \dots + Z_{km_k}t^{m_k-1}]e^{\lambda_k t}.$$

The matrices Z_{kj} are completely determined by A [29]. In particular, if the minimal polynomial has only simple roots, the Lagrange-Sylvester interpolation formula gives for $A \in M(n, \mathbb{C})$ [29]

$$e^{At} = \sum_{k=1}^n \frac{(A - \lambda_1 \mathbb{I}_n) \cdots (A - \lambda_{k-1} \mathbb{I}_n) \cdots (A - \lambda_{k+1} \mathbb{I}_n) \cdots (A - \lambda_n \mathbb{I}_n)}{(\lambda_k - \lambda_1) \cdots (\lambda_k - \lambda_{k-1}) \cdots (\lambda_k - \lambda_{k+1}) \cdots (\lambda_k - \lambda_n)} e^{\lambda_k t}.$$

In the situation of Remark 4, we diagonalize the matrix h via (7.32) and we make a change of variables

$$(7.39) \quad \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

The system (7.20) in the new variables reads

$$\dot{X}' = \Lambda_1 X'; \quad \dot{Y}' = \Lambda_2 Y',$$

with the solution

$$X' = e^{t\Lambda_1} X'_0; \quad Y' = e^{t\Lambda_2} Y'_0.$$

We calculate $W' = X'Y'^{-1}$ and get the expression (7.35b), then we calculate $W'_0 = X'_0(Y'_0)^{-1}$ and we obtain (7.35c), and finally, the equation (7.35a). In the situation of Remark 4, $\Lambda_1 = id$, $\Lambda_2 = -id$.

The assumption contained in (7.36) with the consequence (7.37) is expressed in Proposition 3.1.18 in [1] and (7.38) appears in Remark 1, e). Then we apply the transform (2.11) and the assertion for $\mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$ follows. \square

d. Finally, we discuss **how to solve the decoupled system at c) in Proposition 6**. In (7.16) we introduce $\eta = \xi - i\zeta$, $\xi, \zeta \in \mathbb{R}^n$, we put $\epsilon = b + ia$, where $a, b \in \mathbb{R}^n$. The first order complex differential equation (7.16) is equivalent with a system of first order real differential equations with real coefficients, which we write as

$$(7.40) \quad \dot{Z} = h_r Z + F, \quad Z = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \quad F = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Because of (7.11), the matrices p, m, n are symmetric, while q is antisymmetric, as in (7.12). This means that the matrix h_r has the form (2.9), i.e. $h_r \in \mathfrak{sp}(n, \mathbb{R})$, as was already underlined in Remark 3. With general considerations on solving first order linear systems of nonhomogenous equations [31] particularized to the case of Hamiltonian matrices [46], we get

Remark 6. Let $\Delta(t, t_0)$ be the fundamental solution of the homogeneous equation $\dot{Z} = h_r Z$ associated with (7.40), where $h_r \in \mathfrak{sp}(n, \mathbb{R})$. The fundamental matrix solution $\Delta(t, t_0)$ of the linear Hamiltonian system is symplectic. The solution of the inhomogeneous equation (7.40) is

$$Z(t) = \Delta(t, t_0)Z(t_0) + \int_{t_0}^t \Delta(t, \tau)F(\tau) d\tau.$$

In the case of constant coefficients, the solution of (7.40) is

$$Z = e^{h_r(t-t_0)} Z_0 + \int_{t_0}^t e^{h_r(t-\tau)} F(\tau) d\tau.$$

So, solving the equation (7.40), we find $\eta = \xi - i\zeta$, and we find the solution of (7.16).

The solution of (7.13b) with coefficients (7.15), i.e. (7.17b), is obtained via the *FC* transform $z = \eta - W\bar{\eta}$.

8. PHASES

8.1. Berry phase for \mathcal{D}_n^J . Formula (7.9e) for the Berry phase on the Siegel-Jacobi ball \mathcal{D}_n^J in the variables $(W, z) \in \mathcal{D}_n \times \mathbb{C}^n$ reads:

$$\frac{2}{i} d\varphi_B = \left(\sum d w_{ij} \frac{\partial}{\partial w_{ij}} - cc \right) f + \left(\sum d z_i \frac{\partial}{\partial z_i} - cc \right) f,$$

where $(X - cc)$ means $(X - \bar{X})$. Above f is the Kähler potential (3.11) written as

$$f = -\frac{k}{2} \log(\det(\mathbb{I}_n - W\bar{W})) + F.$$

With (4.19), (4.22) and (4.17), we have

$$(8.1a) \quad \frac{\partial f}{\partial z_i} = \bar{\eta}_i,$$

$$(8.1b) \quad \frac{\partial f}{\partial w_{ik}} = \frac{k}{2}(2X_{ik} - X_{ik}\delta_{ik}) + \bar{\eta}_i\bar{\eta}_k - \frac{1}{2}\bar{\eta}_i\bar{\eta}_k\delta_{ik}, \quad \text{or} \quad \nabla f = \frac{1}{2}(kX + \bar{\eta} \otimes \bar{\eta}).$$

With the *FC*-transform (6.1) $z_i = \eta_i - w_{ij}\bar{\eta}_j$, we find

Remark 7. *The Berry phase on the Siegel-Jacobi ball \mathcal{D}_n^J is expressed in the variables $(W, \eta) \in \mathcal{D}_n \times \mathbb{C}^n$ as*

$$\begin{aligned} \frac{2}{i} d\varphi_B &= \left\{ \left[\sum \frac{k}{2}(2X_{ij} - X_{ij}\delta_{ij}) - \frac{1}{2}\bar{\eta}_i\bar{\eta}_j\delta_{ij} \right] d w_{ij} - cc \right\} + [(\bar{\eta}_i + \bar{w}_{ij}\eta_j) d \eta_i - cc] \\ &= \left\{ \frac{k}{2}[2\text{Tr}(X d W) - \text{Tr}(\text{diag}(X)\text{diag}(d W))] - \frac{1}{2}\bar{\eta}^t \text{diag}(d W)\bar{\eta} - cc \right\} \\ &\quad + [d \bar{\eta}^t(\bar{\eta} + \bar{W}\eta) - cc]. \end{aligned}$$

8.2. Dynamical phase. We calculate the energy function attached to the Hamiltonian (7.10) using the formula (7.9b). We write down $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, where \mathcal{H}_1 is the first term $\mathcal{H}_1 = \sum_{\lambda \in \Delta} \epsilon_\lambda \tilde{P}_\lambda$ in (7.9c), while \mathcal{H}_2 is the rest in (7.9c), which corresponds to $i \sum_{\beta \in \Delta_+} \tilde{z}_\beta \partial_\beta f$, but we prefer the brute force calculation with the formulae (4.6). With (4.6) and (8.1a), we get

$$(8.2) \quad \mathcal{H}_1 = \epsilon^t z + \frac{k}{4} \text{Tr}(\epsilon_0) + \frac{k}{2} \text{Tr}(\epsilon_- W) + \frac{1}{2} z^t \epsilon_- z.$$

For \mathcal{H}_2 , we get with (4.6)

$$(8.3a) \quad \mathcal{H}_2 = \epsilon^t W \bar{\eta} + \bar{\epsilon}^t \bar{\eta} + \frac{1}{2} z^t \epsilon_0^t \bar{\eta} + z^t \epsilon_- W \bar{\eta} + \mathcal{H}_3,$$

$$(8.3b) \quad \mathcal{H}_3 = (\epsilon_{kl}^0 w_{li} \nabla_{ik} + \epsilon_{kl}^+ \nabla_{kl} + \epsilon_{kl}^- w_{\alpha l} w_{ki} \nabla_{i\alpha}) f,$$

where ∇f has the value (8.1b).

Let us denote by $\Lambda = \Lambda(W, \epsilon_-, \epsilon_+, \epsilon_0)$ the matrix appearing in the rhs of (7.17a), i.e. $\Lambda = iQ$. We get

$$(8.4) \quad \mathcal{H}_3 = \frac{1}{2}[k\text{Tr}(\tilde{\Lambda}X) + \bar{\eta}^t \tilde{\Lambda} \bar{\eta}], \quad \text{where } \tilde{\Lambda} = \Lambda(W, \epsilon_+, \epsilon_-, \epsilon_0^t) = \epsilon_+ + (\epsilon_0 W)^s + W\epsilon_- W.$$

Now we apply the FC -transform (6.1) in (8.2) and (8.3) and we express the energy function as sum of two separated terms in the independent variables $\eta \in \mathbb{C}^n$ and $W \in \mathcal{D}_n$

$$(8.5a) \quad \mathcal{H} = \mathcal{H}_\eta + \mathcal{H}_w,$$

$$(8.5b) \quad \mathcal{H}_\eta = \epsilon^t \eta + \bar{\epsilon}^t \bar{\eta} + \frac{1}{2}(\eta^t \epsilon_- \eta + \bar{\eta}^t \epsilon_+ \bar{\eta} + \bar{\eta}^t \epsilon_0 \eta),$$

$$(8.5c) \quad \mathcal{H}_w = \frac{k}{2}\text{Tr}\{(\epsilon_0)^s + [W\epsilon_- + \epsilon_+ \bar{W} + (\epsilon_0 W)^s \bar{W}](\mathbb{I}_n - W\bar{W})^{-1}\}.$$

We calculate the critical points of the energy function (8.5) attached to linear hermitian Hamiltonian (7.10). We get

$$(8.6a) \quad \nabla \mathcal{H}_w = 2(\mathbb{I}_n - \bar{W}W)^{-1} \bar{\tilde{\Lambda}}(\mathbb{I}_n - W\bar{W})^{-1},$$

$$(8.6b) \quad \frac{\partial \mathcal{H}_\eta}{\partial \eta} = \epsilon + \epsilon_- \eta + \frac{1}{2} \epsilon_0^t \bar{\eta}.$$

We have proved

Remark 8. *The energy function in the variables $(\eta, W) \in \mathbb{C}^n \times \mathcal{D}_n$ attached to the linear hermitian Hamiltonian (7.10) is given by (8.5). The critical points W_c of the energy functions (8.5) are the solutions of the algebraic matrix Riccati equations $\bar{\tilde{\Lambda}} = 0$, i. e. the solutions of the equation $\dot{W} = 0$ in (7.17a). The critical value η_c corresponds to the solution $\dot{\xi} = 0$, $-\dot{\zeta} = 0$ in (7.40).*

Acknowledgement. The question of the fundamental conjecture for the Siegel-Jacobi disk was raised to me by Pierre Bieliavsky. I am grateful to Mircea Bundaru for useful discussions and for bringing into my attention formula (4.5). This investigation was supported by the ANCS project program PN 09 37 01 02/2009 and by the UEFISCDI-Romania program PN-II Contract No. 55/05.10.2011.

REFERENCES

- [1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, The Benjamin/Cummings publishing company, INC, Reading Massachusetts, London, 1978.
- [2] S. T. Ali, J.-P. Antoine and J.-P. Gazeau, *Coherent states, wavelets, and their generalizations*, Springer-Verlag, New York, 2000.
- [3] V. Bargmann, Group representations on Hilbert spaces of analytic functions, in *Analytic methods in Mathematical Physics*, R.P Gilbert and R.G. Newton, Editors, Gordon and Breach, Science publishers, New York, London, Paris, 1970.
- [4] S. Berceanu and A. Gheorghe, *On equations of motion on Hermitian symmetric spaces*, J. Math. Phys. **33** (1992) 998-1007.
- [5] S. Berceanu and L. Boutet de Monvel, *Linear dynamical systems, coherent state manifolds, flows and matrix Riccati equation*, J. Math. Phys. **34** (1993) 2353-2371.
- [6] S. Berceanu and A. Gheorghe, *Differential operators on orbits of coherent states*, Romanian J. Phys. **48** (2003) 545-556; arXiv: math.DG/0211054.

- [7] S. Berceanu, Realization of coherent state algebras by differential operators, in *Advances in Operator Algebras and Mathematical Physics*, F. Boca, O. Bratteli, R. Longo and H. Siedentop, Editors, The Theta Foundation, Bucharest 1–24, 2005.
- [8] S. Berceanu, *A holomorphic representation of the Jacobi algebra*, Rev. Math. Phys. **18** (2006) 163–199.
- [9] S. Berceanu, *Coherent states associated to the Jacobi group - a variation on a theme by Erich Kähler*, J. Geom. Symmetry Physics **9** (2007) 1–8.
- [10] S. Berceanu, A holomorphic representation of Jacobi algebra in several dimensions, in *Perspectives in Operator Algebra and Mathematical Physics*, F.-P. Boca, R. Purice and S. Stratila eds, The Theta Foundation, Bucharest 1–25, 2008; arXiv: math.DG/060404381.
- [11] S. Berceanu and A. Gheorghe, *Applications of the Jacobi group to Quantum Mechanics*, Romanian J. Physics **53** (2008) 1013–1021; arXiv: 0812.0448 (math.DG).
- [12] S. Berceanu and A. Gheorghe, *On the geometry of Siegel-Jacobi domains*, Int. J. Geom. Methods Mod. Phys. **8** (2011) 1783–1798; arXiv:1011.3317v1 (2010).
- [13] S. Berceanu, Classical and quantum evolution on the Siegel-Jacobi manifolds, in *Geometric Methods in Physics, XXX Workshop 2011*, Trends in Mathematics, eds. P. Kielanowski, S. T. Ali, A. Odziejewicz, M. Schlichenmaier and Th. Voronov Birkhäuser, Springer Basel AG, 2013 pp. 43–52.
- [14] S. Berceanu, *Consequences of the fundamental conjecture for the motion on the Siegel-Jacobi disk*, Int. J. Geom. Methods Mod. Phys. **10** (2013) 1250076 (18 pp); arXiv: 1110.5469v2 [math.DG] 23 Apr 2012.
- [15] F. A. Berezin, *Quantization in complex symmetric spaces*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975) 363–402, 472.
- [16] F. A. Berezin, *The general concept of quantization*, Commun. Math. Phys. **40** (1975) 153–174.
- [17] F. A. Berezin, *Models of Gross-Neveu type are quantization of a Classical Mechanics with a nonlinear phase space*, Commun. Math. Phys. **63** (1978) 131–153.
- [18] R. Berndt, Sur l'arithmétique du corps des fonctions elliptiques de niveau N , in *Seminar on number theory, Paris 1982–83*, Progr. Math. **51** Birkhäuser, Boston, MA (1984) 21–32.
- [19] R. Berndt and S. Böcherer, *Jacobi forms and discrete series representations of the Jacobi group*, Math. Z. **204** (1990) 13–44.
- [20] R. Berndt and R. Schmidt, *Elements of the representation theory of the Jacobi group*, Progress in Mathematics **163** Birkhäuser Verlag, Base, 1998.
- [21] A. Borel, *Kählerian coset spaces of semi-simple Lie groups*, Proc. Nat. Acad. Sci. USA **40** (1954) 1147–1151.
- [22] J. Cogdell, S. Gindikin and P. Sarnak, Editors, *Selected works of Ilya Piatetski-Shapiro*, American Mathematical Society, Providence, Rhode Island, 2000.
- [23] V. V. Dodonov, I. A. Malkin and V. I. Man'ko, *Even and odd coherent states and excitations of a singular oscillator* Physica **72** (1974) 597–615.
- [24] J. Dorfmeier and K. Nakajima, *The fundamental conjecture for homogeneous Kähler manifolds*, Acta Mathematica **161** (1988) 23–70.
- [25] P. D. Drummond and Z. Ficek, Editors, *Quantum Squeezing*, Springer, Berlin, 2004.
- [26] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics **55** Birkhäuser, Boston, MA, 1985.
- [27] R. Feynman, *An Operator Calculus Having Applications in Quantum Electrodynamics*, Phys. Rev. **84** (1951) 108–128.
- [28] G. B. Folland, *Harmonic analysis in phase space*, Princeton University Press, Princeton, New Jersey, 1989.
- [29] F. R. Gantmacher, *The Theory of matrices*, Volume one, AMS Chelsea Publishing, American Mathematical Society Providence, Rhode Island, 2000.
- [30] C. R. Hagen, *Scale and conformal transformations in Galilean-covariant field theory*, Phys. Rev. D **5** (1972) 377–388.
- [31] P. Hartman, *Ordinary differential equations*, New York, John Wiley, 1964.

- [32] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.
- [33] J. N. Hollenhorst, *Quantum limits on resonant-mass gravitational-wave detectors*, Phys. Rev. D **19** (1979) 1669–1679.
- [34] L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, Translated from Russian by Leo Ebner and Adam Korányi, Amer. Math. Soc., Providence, R.I., 1963.
- [35] K. Husimi, *Miscellaneous in Elementary Quantum Mechanics*, II Prog. Theor. Phys. **9** (1953) 381–402.
- [36] E. Kähler, *Raum-Zeit-Individuum*, Rend. Accad. Naz. Sci. XL Mem. Mat. **5** 16 (1992) 115–177.
- [37] *Erich Kähler: Mathematische Werke; Mathematical Works*, R. Berndt and O. Riemenschneider, Editors, Walter de Gruyter, Berlin-New York, 2003.
- [38] E. H. Kennard, *Zur Quantenmechanik einfacher Bewegungstypen*, Zeit. Phys. **44** (1927) 326–352.
- [39] P. Kramer and M. Saraceno, *Semicoherent states and the group $\mathrm{ISp}(2, \mathbb{R})$* , Physics **114A** (1982) 448–453.
- [40] A. J. Laub and K. Meyer, *Canonical forms of symplectic and Hamiltonian matrices*, Celes. Mech. **9** (1974) 213–238.
- [41] M. H. Lee, *Theta functions on hermitian symmetric domains and Fock representations*, J. Aust. Math. Soc. **74** (2003) 201–234.
- [42] J. J. Levin, *On the matrix Riccati equation*, Proc. Am. Math. Soc. **10** (1959) 519–524.
- [43] E. Y. C. Lu, *New coherent states of the electromagnetic field*, Lett. Nuovo. Cimento **2** (1971) 1241–1244.
- [44] L. Mandel and E. Wolf, *Optical coherence and quantum optics*, Cambridge University Press, 1995.
- [45] Y. Matsushima, *Sur les espaces homogènes Kähleriens d'un group de Lie reductif*, Nagoya Math. J. **11** (1957) 53–60.
- [46] K. R. Meyer, G. R. Hall and D. Offin, *Introduction to Hamiltonian dynamical systems and the N-body problem*, Springer, 2009.
- [47] U. Niederer, *The maximal kinematical invariance group of the harmonic oscillator*, Helv. Phys. Acta **46** (1973) 191–200.
- [48] A. M. Perelomov, *Generalized Coherent States and their Applications*, Springer, Berlin, 1986.
- [49] C. Quesne, *Vector coherent state theory of the semidirect sum Lie algebras $\mathrm{wsp}(2N, \mathbb{R})$* , J. Phys. A: Gen. **23** (1990) 847–862.
- [50] I. Satake, *Algebraic structures of symmetric domains*, Publ. Math. Soc. Japan **14**, Princeton Univ. Press, 1980.
- [51] J. Schwinger, *The Theory of Quantized Fields. III.*, Phys. Rev. **91** (1953) 728–740.
- [52] A. Shapere and F. Wilczek, Editors, *Geometric Phases in Physics*, World Scientific, Singapore, 1989.
- [53] K. Shuman, *Complete signal processing bases and the Jacobi group*, J. Math. Anal. Appl. **278** (2003), 203–213.
- [54] C. L. Siegel, *Symplectic geometry*, Academic Press, New York, 1964.
- [55] S. Sivakumar, *Studies on nonlinear quantum optics*, J. Opt. B Quantum Semiclass. Opt. **2**, R61–R75 (2000).
- [56] P. Stoler, *Equivalence classes of minimum uncertainty packets*, Phys. Rev. D **1** (1970) 3217–3219.
- [57] K. Takase, *A note on automorphic forms*, J. Reine Angew. Math. **409** (1990) 138–171.
- [58] K. Takase, *On unitary representations of Jacobi groups*, J. Reine Angew. Math. **430** (1992) 130–149.
- [59] K. Takase, *On Siegel modular forms of half-integral weights and Jacobi forms*, Trans. Amer. Math. Soc. **351** (1999) 735–780.
- [60] E. B. Vinberg and S. G. Gindikin, *Kählerian manifolds admitting a transitive solvable automorphism group*, Math. Sb. **74** (116) (1967) 333–351.
- [61] H. C. Wang, *Closed manifolds with complex structure*, Amer. J. Math. **76** (1954) 1–32.

- [62] K. B. Wolf, The Heisenberg-Weyl ring in quantum mechanics, in *Group theory and its applications*, **3**, E. M. Loebl, Editor, Academic Press, New York, 189–247, 1975.
 - [63] V. A. Yakulovich and V. M. Starzhinskii, *Linear differential equations with periodic coefficients*, John Wiley & Sons, NewYork, Toronto, 1975.
 - [64] J.-H. Yang, *The method of orbits for real Lie groups*, Kyungpook Math. J. **42** (2002) 199–272.
 - [65] J.-H. Yang, *Invariant metrics and Laplacians on the Siegel-Jacobi spaces*, J. Number Theory, **127** (2007) 83–102.
 - [66] J.-H. Yang, *A partial Cayley transform for Siegel-Jacobi disk*, J. Korean Math. Soc. **45** (2008) 781–794.
 - [67] J.-H. Yang, *Remark on harmonic analysis on Siegel-Jacobi space*, arXiv: math/0612230v3 [math.NT]; 2009.
 - [68] J.-H. Yang, *Invariant metrics and Laplacians on the Siegel-Jacobi disk*, Chin. Ann. Math. **31B** (2010) 85–100.
 - [69] H. P. Yuen, *Two-photon coherent states of the radiation field*, Phys. Rev. A **13** (1976) 2226–2243.
- March 3, 2013

(Stefan Berceanu) HORIA HULUBEI NATIONAL INSTITUTE FOR PHYSICS AND NUCLEAR ENGINEERING, DEPARTMENT OF THEORETICAL PHYSICS, P.O.B. MG-6, 077125 MAGURELE, ROMANIA
E-mail address: Berceanu@theory.nipne.ro